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EFFECTIVE POTENTIAL MODELS FOR HADRONS

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Preface

The aim of these lectures is to give a self-contained introduction to nonrelativistic potential models, to their formulation as well as to their possible applications. At the price of some lack of (in a mathematical sense) rigorous derivations, we try to give a feeling and understanding for the simplest conceivable method to extract the explicit form of the forces acting between quarks from the interplay between experimental observations and theoretical considerations. According to this spirit, we demonstrate, in detail, how to obtain the underlying Hamiltonian and how to determine the Lorentz structure of the quark–(anti-)quark interaction potential from well-established experimental facts.

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Chapter 1

Nonrelativistic Potential

In principle, the appropriate framework for the description of bound states within relativistic quantum field theories is the Bethe–Salpeter formalism. There are, however, some circumstances which are opposed to this. The Bethe–Salpeter equation cannot be solved in general. The interaction kernel entering in this equation is not derivable from QCD either. The propagators of the constituents have to be approximated by their free form, the involved masses, however, being interpreted as effective (“constituent”) ones. So, even if one is willing to put up with the complexity of the Bethe–Salpeter formalism, it is hard to obtain information from this approach.

The alternative which comes closest to one’s physical intuition is the description of bound states with the help of the Schrödinger equation [1, 2, 3]

$$H \psi = E \psi ,$$

where the nonrelativistic Hamiltonian for a quantum system consisting of two particles with masses m_1 and m_2 , respectively, which interact via some potential $V(\mathbf{x})$ is given in the center-of-momentum frame by

$$H = m_1 + m_2 + \frac{\mathbf{p}^2}{2\mu} + V(\mathbf{x}) ;$$

here μ denotes the reduced mass,

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2} .$$

Our main task is simply to find that potential $V(\mathbf{x})$ which describes the interaction of the two particles constituting the bound state under consideration. By investigating the corresponding scattering problem of the involved bound-state constituents, the *perturbatively* accessible part of this potential may be derived according to the following recipe (for details see, for instance, Refs. [4, 5]):

1. Compute the scattering amplitude T_{fi} , which is defined in terms of the S-matrix element S_{fi} introduced in Appendix A,

$$S_{fi} \equiv \langle f, \text{out} | i, \text{in} \rangle ,$$

by the decomposition

$$S_{fi} = \delta_{fi} + i (2\pi)^4 \delta^{(4)}(P_f - P_i) T_{fi} ,$$

for the *elastic* scattering process $i \rightarrow f$ in lowest non-trivial order of perturbation theory, the so-called “first Born approximation.”

2. Perform the nonrelativistic limit, realized by the vanishing of the momenta \mathbf{p} of the involved bound-state constituents; we indicate this limit rather symbolically by

$$\mathbf{p} \rightarrow 0 .$$

3. Obtain the configuration-space interaction potential sought after, $V(\mathbf{x})$, as the Fourier transform of the above scattering amplitude T_{fi} :

$$V(\mathbf{x}) = -(2\pi)^3 \int d^3k \exp(-i \mathbf{k} \cdot \mathbf{x}) T_{fi}(k) .$$

For the sake of simplicity, we split off all the normalization factors of the one-particle wave functions, given for a fermion of mass m and kinetic energy

$$E_p = \sqrt{\mathbf{p}^2 + m^2}$$

by

$$\frac{1}{(2\pi)^{3/2}} \sqrt{\frac{m}{E_p}} ;$$

we thereby define a quantity t according to

$$T_{fi} =: \frac{1}{(2\pi)^6} \frac{m^2}{\sqrt{E_{p_1} E_{p_2} E_{q_1} E_{q_2}}} t .$$

In the framework of this nonrelativistic treatment, the short-range part of the quark–antiquark potential (which is of perturbative origin!) will be determined from quantum chromodynamics (QCD) according to the above prescription. The shape of the long-range, confining part of the quark–antiquark potential (which is of nonperturbative origin!) will be obtained from the analysis of the possible Lorentz structures of the potential, its coordinate dependence from the comparison of some of the resulting predictions with experiment.

1.1 Nonrelativistic limit

In the *nonrelativistic limit*, our whole formalism, so to say, “collapses” to an extremely simple one:

- The relativistically correct expression for the one-particle kinetic energy,

$$E_p = \sqrt{\mathbf{p}^2 + m^2} ,$$

reduces to

$$E_p = m .$$

- The Dirac spinors $u(p, \sigma)$ and $v(p, \sigma)$ describing fermions of mass m , four-momentum p , and spin polarization σ ,

$$u(p, \sigma) = \sqrt{\frac{E_p + m}{2m}} \begin{pmatrix} 1 \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E_p + m} \end{pmatrix} \chi_\sigma ,$$

$$v(p, \sigma) = \sqrt{\frac{E_p + m}{2m}} \begin{pmatrix} \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E_p + m} \\ 1 \end{pmatrix} \chi_\sigma^c , \quad \chi_\sigma^c \equiv -i \sigma_2 \chi_\sigma^* ,$$

where $\boldsymbol{\sigma} \equiv \{\sigma_i, i = 1, 2, 3\}$ are the three Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

reduce to the nonrelativistic spinors

$$u(\sigma) = \begin{pmatrix} \chi_\sigma \\ 0 \end{pmatrix} ,$$

$$v(\sigma) = \begin{pmatrix} 0 \\ \chi_\sigma^c \end{pmatrix} .$$

- The normalization of the Dirac spinors adopted by us,

$$u^\dagger(p, \sigma) u(p, \tau) = v^\dagger(p, \sigma) v(p, \tau) = \frac{E_p}{m} \delta_{\sigma\tau} ,$$

reduces to the nonrelativistic normalization

$$u^\dagger(\sigma) u(\tau) = \chi_\sigma^\dagger \chi_\tau = \delta_{\sigma\tau} ,$$

$$v^\dagger(\sigma) v(\tau) = \chi_\sigma^{c\dagger} \chi_\tau^c = \delta_{\tau\sigma} .$$

1.2 Static potential in quantum electrodynamics

Let us illustrate the very simple procedure outlined above by applying it first to electron–positron scattering:

$$e^-(p_1, \sigma_1) + e^+(p_2, \sigma_2) \rightarrow e^-(q_1, \tau_1) + e^+(q_2, \tau_2) .$$

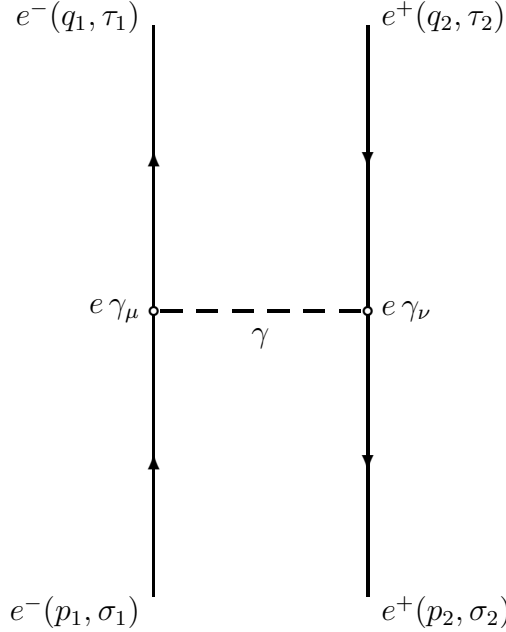


Figure 1.1: Electron–positron scattering, one-photon exchange graph.

The interaction term in the Lagrangian of quantum electrodynamics (QED) for the coupling of a fermion with electric charge e , described by the Dirac spinor field $\psi(x)$, to the photon field $A^\mu(x)$ is well known:

$$\mathcal{L}_I^{\text{QED}}(x) = e \bar{\psi}(x) \gamma_\mu \psi(x) A^\mu(x) .$$

In lowest non-trivial order of the perturbative loop expansion, just two Feynman diagrams contribute to the scattering amplitude T_{fi} for elastic electron–positron scattering:

- the exchange of a single photon, γ , between electron and positron, as depicted in Fig. 1.1; and
- the annihilation of the electron–positron pair into a single photon, γ , followed by a subsequent creation of an electron–positron pair by this single photon, as depicted in Fig. 1.2.

With the help of the Feynman rules given for an arbitrary, that is, in general, non-Abelian, gauge theory in Appendix B, it's straightforward to find the corresponding scattering amplitude T_{fi} :

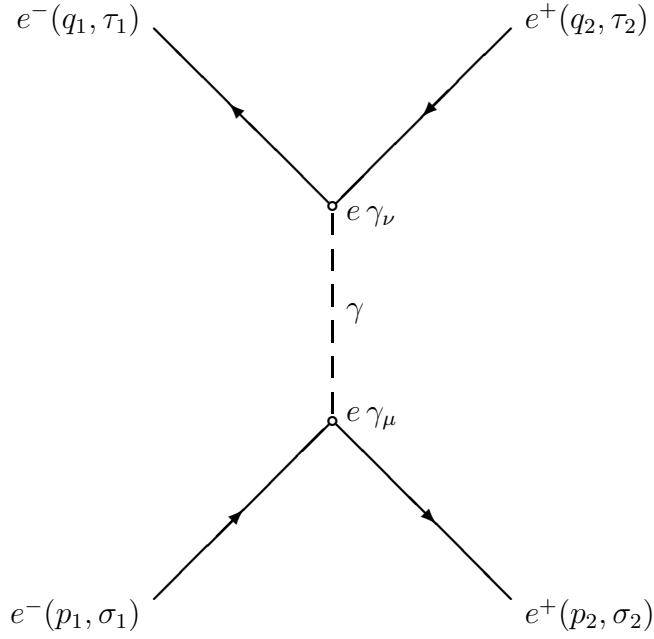


Figure 1.2: Electron–positron scattering, pair annihilation graph.

- The contribution of the one-photon exchange graph in Fig. 1.1 to our scattering amplitude t reads

$$t_{\text{exch}} = -\frac{e^2}{k^2} \bar{u}(q_1, \tau_1) \gamma_\mu u(p_1, \sigma_1) \bar{v}(p_2, \sigma_2) \gamma^\mu v(q_2, \tau_2) , \quad (1.1)$$

where k denotes the involved momentum transfer,

$$k \equiv p_1 - q_1 = q_2 - p_2 .$$

The square of this momentum transfer,

$$\begin{aligned} k^2 &\equiv (p_1 - q_1)^2 \\ &= (E_{p_1} - E_{q_1})^2 - \mathbf{k}^2 , \end{aligned}$$

which enters in the denominator of the scattering amplitude t_{exch} , reduces in the nonrelativistic limit to

$$k^2 = -\mathbf{k}^2 .$$

The spinor factors $\bar{u} \gamma^\mu u$ and $\bar{v} \gamma^\mu v$ may be evaluated very easily:

- For our particular choice for the normalization of the Dirac spinors u and v , we have, in the nonrelativistic limit,

$$\begin{aligned} \bar{u}(\tau_1) \gamma_0 u(\sigma_1) &\equiv u^\dagger(\tau_1) u(\sigma_1) = \delta_{\tau_1 \sigma_1} , \\ \bar{v}(\sigma_2) \gamma_0 v(\tau_2) &\equiv v^\dagger(\sigma_2) v(\tau_2) = \delta_{\tau_2 \sigma_2} . \end{aligned}$$

- In the Dirac representation, the Dirac matrices $\gamma^\mu \equiv \{\gamma^0, \boldsymbol{\gamma}\}$ are explicitly given by

$$\begin{aligned}\gamma_0 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \\ \boldsymbol{\gamma} &= \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} .\end{aligned}$$

Inserting these explicit representations of the Dirac matrices, we obtain, in the nonrelativistic limit,

$$\begin{aligned}\bar{u}(\tau_1) \boldsymbol{\gamma} u(\sigma_1) &\equiv u^\dagger(\tau_1) \gamma_0 \boldsymbol{\gamma} u(\sigma_1) \\ &= u^\dagger(\tau_1) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} u(\sigma_1) \\ &= u^\dagger(\tau_1) \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} u(\sigma_1) \\ &= (\chi_{\tau_1}^\dagger, 0) \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \begin{pmatrix} \chi_{\sigma_1} \\ 0 \end{pmatrix} \\ &= 0\end{aligned}$$

and, similarly,

$$\bar{v}(\sigma_2) \boldsymbol{\gamma} v(\tau_2) = 0 .$$

Accordingly, the scattering amplitude t_{exch} of Eq. (1.1) reduces to

$$t_{\text{exch}} = \frac{e^2}{\mathbf{k}^2} \delta_{\tau_1 \sigma_1} \delta_{\tau_2 \sigma_2} .$$

- The contribution of the pair annihilation graph in Fig. 1.2 to our scattering amplitude t reads

$$t_{\text{ann}} = \frac{e^2}{(p_1 + p_2)^2} \bar{u}(q_1, \tau_1) \gamma_\mu v(q_2, \tau_2) \bar{v}(p_2, \sigma_2) \gamma^\mu u(p_1, \sigma_1) .$$

Here, the total momentum P of the system under consideration,

$$P \equiv p_1 + p_2 = q_1 + q_2 ,$$

enters in the denominator of the scattering amplitude t_{ann} . In that case, however, the square of this total momentum,

$$\begin{aligned}P^2 &\equiv (p_1 + p_2)^2 \\ &= (E_{p_1} + E_{p_2})^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2 ,\end{aligned}$$

reduces in the nonrelativistic limit to

$$P^2 \equiv (p_1 + p_2)^2 = (2m)^2 .$$

Thus, compared with the contribution to the scattering amplitude t arising from one-photon exchange, t_{exch} , the contribution to the scattering amplitude t arising from pair annihilation, t_{ann} , will be of the order of

$$\frac{\mathbf{k}^2}{m^2} .$$

This observation indicates that the annihilation contribution t_{ann} represents, in any case, already some relativistic correction to the exchange contribution t_{exch} , for which there will be no room at all within a purely nonrelativistic investigation and which, therefore, has to be neglected for the present discussion.

We are unambiguously led to the conclusion that, in the nonrelativistic limit, only the one-photon exchange graph contributes to the T-matrix element for elastic electron-positron scattering:

$$T_{\text{fi}} = \frac{1}{(2\pi)^6} t = \frac{1}{(2\pi)^6} \frac{e^2}{\mathbf{k}^2} \delta_{\tau_1 \sigma_1} \delta_{\tau_2 \sigma_2} .$$

According to step 3 of our procedure, the interaction potential $V(\mathbf{x})$ is obtained as the Fourier transform of the T-matrix element T_{fi} . Since, at present, we are exclusively interested in the nonrelativistic limit, we shall obtain in this way only the nonrelativistic (or static) part $V_{\text{NR}}(\mathbf{x})$ of the potential:

$$\begin{aligned} V_{\text{NR}}(\mathbf{x}) &= -(2\pi)^3 \int d^3k \exp(-i \mathbf{k} \cdot \mathbf{x}) T_{\text{fi}}(k) \\ &= -\frac{1}{(2\pi)^3} \int d^3k \exp(-i \mathbf{k} \cdot \mathbf{x}) t \\ &= -\frac{1}{(2\pi)^3} e^2 \int d^3k \frac{\exp(-i \mathbf{k} \cdot \mathbf{x})}{\mathbf{k}^2} . \end{aligned}$$

The result of the required integration may immediately be written down:

1. The integral is obviously invariant under rotations. Consequently, it has to be some function Φ of the radial coordinate $r \equiv |\mathbf{x}|$ only:

$$\int d^3k \frac{\exp(-i \mathbf{k} \cdot \mathbf{x})}{\mathbf{k}^2} = \Phi(r) .$$

2. For dimensional reasons, this function $\Phi(r)$ has to be proportional to the inverse of r :

$$\Phi(r) \propto \frac{1}{r} .$$

These considerations justify the ansatz

$$\frac{1}{(2\pi)^3} \int d^3k \frac{\exp(-i \mathbf{k} \cdot \mathbf{x})}{\mathbf{k}^2} = \frac{A}{r} ,$$

with some dimensionless constant A . We determine the constant A by applying the Laplacian $\Delta \equiv \nabla \cdot \nabla$ to both sides of this ansatz:

- For the left-hand side, we find

$$\begin{aligned} \frac{1}{(2\pi)^3} \Delta \int d^3k \frac{\exp(-i \mathbf{k} \cdot \mathbf{x})}{\mathbf{k}^2} &= -\frac{1}{(2\pi)^3} \int d^3k \exp(-i \mathbf{k} \cdot \mathbf{x}) \\ &= -\delta^{(3)}(\mathbf{x}) . \end{aligned}$$

- For the right-hand side, upon remembering the relation

$$\Delta \frac{1}{r} = -4\pi \delta^{(3)}(\mathbf{x}) ,$$

we find

$$A \Delta \frac{1}{r} = -4\pi A \delta^{(3)}(\mathbf{x}) .$$

By comparison, the dimensionless proportionality factor, A , is pinned down to the value

$$A = \frac{1}{4\pi} .$$

With due satisfaction, we realize that, by following step by step our general prescription given in our introductory remarks to this chapter, one is indeed able to recover, from the nonrelativistic limit of the Born approximation to the T-matrix element for (elastic) electron–positron scattering, the static Coulomb potential of quantum electrodynamics:

$$V_{\text{NR}}^{\text{QED}}(r) = -\frac{e^2}{4\pi r} ,$$

or, with the usually employed definition

$$\alpha_{\text{em}} \equiv \frac{e^2}{4\pi}$$

of the electromagnetic fine structure constant,

$$\boxed{V_{\text{NR}}^{\text{QED}}(r) = -\frac{\alpha_{\text{em}}}{r}} .$$

1.3 Static potential in quantum chromodynamics

The overwhelming success in the case of quantum electrodynamics has contributed to enhance our confidence in our prescription of extracting the (perturbatively accessible part of an) effective interaction potential from the relevant elastic-scattering problem. Hence, we do not hesitate to apply this procedure also to the case of quantum chromodynamics.

The relevant situation for the determination of the potential which describes the quark forces acting within mesons is the quark–antiquark scattering

$$q_i(p_1, \sigma_1) + \bar{q}_j(p_2, \sigma_2) \rightarrow q_k(q_1, \tau_1) + \bar{q}_\ell(q_2, \tau_2) ,$$

where the indices $i, j, \dots = 1, 2, 3$ denote the colour degrees of freedom of the involved quarks.

According to our brief but nevertheless comprehensive—not to say, exhaustive—sketch of quantum chromodynamics given in Appendix C, the coupling, with the interaction strength g_s , of a quark q , represented by the Dirac spinor field $q_i(x)$, to the gluon fields $G_a^\mu(x)$, $a = 1, 2, \dots, 8$, is described by the interaction Lagrangian

$$\mathcal{L}_I^{\text{QCD}}(x) = g_s \bar{q}_i(x) \gamma_\mu \frac{\lambda_{ij}^a}{2} q_j(x) G_a^\mu(x) ,$$

where λ_a , $a = 1, 2, \dots, 8$, are the eight Gell-Mann matrices; an explicit representation of these matrices may be found in Appendix D.2. They serve to construct a fundamental (three-dimensional) representation of the generators of the gauge group $\text{SU}(3)$ of quantum chromodynamics:

$$T_{\text{fund}}^a = \frac{\lambda_a}{2} .$$

Because of the (structural) similarity of the interaction Lagrangians \mathcal{L}_I of quantum electrodynamics, $\mathcal{L}_I^{\text{QED}}$, and quantum chromodynamics, $\mathcal{L}_I^{\text{QCD}}$, again only two Feynman graphs potentially contribute, in lowest non-trivial order of the perturbative loop expansion, to the scattering amplitude T_{fi} for elastic quark–antiquark scattering:

- the exchange of a single gluon, G_a , between quark and antiquark, as depicted in Fig. 1.3; and
- the “annihilation” of the quark–antiquark pair into a single gluon, G_a , followed by the subsequent creation of a quark–antiquark pair by this single gluon, as depicted in Fig. 1.4.

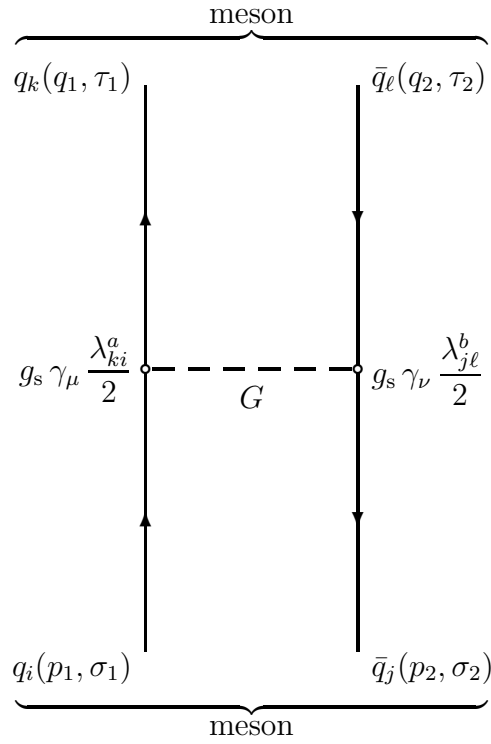


Figure 1.3: Quark–antiquark scattering, one-gluon exchange graph.

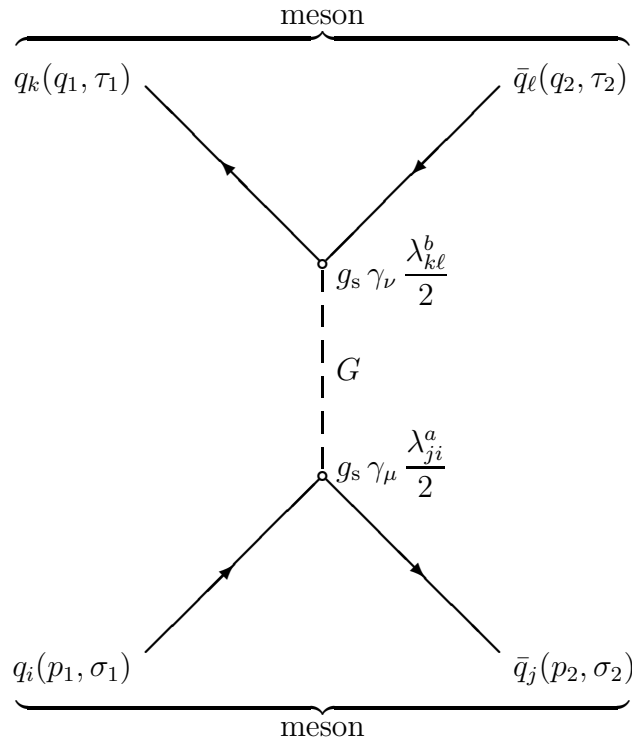


Figure 1.4: Quark–antiquark scattering, pair annihilation graph.

There is absolutely no need to calculate the corresponding T-matrix elements T_{fi} once more. Comparing the above interaction Lagrangians \mathcal{L}_I of quantum electrodynamics, $\mathcal{L}_I^{\text{QED}}$, and quantum chromodynamics, $\mathcal{L}_I^{\text{QCD}}$, we realize that—loosely speaking—we may obtain the transition amplitudes required in the present case from the ones computed in the previous section by simply replacing in the latter the electric charge e by the expression

$$g_s \frac{\lambda^a}{2} .$$

However, we have to take into account that, according to the famous confinement hypothesis, all the quarks inside a hadron form necessarily a colour-singlet state. Consequently, we feel obliged to amend—which, as there is no danger of confusion, we do without change of notations—the transition amplitudes T_{fi} for the scattering of free particles by the (normalized) meson colour wave functions

$$\frac{1}{\sqrt{3}} \delta_{ij} ,$$

which means, in fact, nothing else but an appropriate average over the colour degrees of freedom.

With this proviso, the T-matrix element T_{fi} in question is found as follows:

- The contribution of the one-gluon exchange graph in Fig. 1.3 to our scattering amplitude t reads

$$t_{\text{exch}} = -\frac{g_s^2}{k^2} \frac{\lambda_{ki}^a}{2} \frac{\lambda_{j\ell}^a}{2} \bar{u}(q_1, \tau_1) \gamma_\mu u(p_1, \sigma_1) \bar{v}(p_2, \sigma_2) \gamma^\mu v(q_2, \tau_2) ,$$

where k denotes again the involved momentum transfer,

$$k \equiv p_1 - q_1 = q_2 - p_2 ,$$

or, after the announced multiplication by those meson colour wave functions,

$$\begin{aligned} t_{\text{exch}} &= -\frac{g_s^2}{k^2} \frac{1}{\sqrt{3}} \delta_{ij} \frac{1}{\sqrt{3}} \delta_{k\ell} \frac{\lambda_{ki}^a}{2} \frac{\lambda_{j\ell}^a}{2} \\ &\times \bar{u}(q_1, \tau_1) \gamma_\mu u(p_1, \sigma_1) \bar{v}(p_2, \sigma_2) \gamma^\mu v(q_2, \tau_2) . \end{aligned}$$

With one of the relations given in Appendix D.3, the colour factor stemming from the exchange graph yields

$$\underbrace{\frac{1}{\sqrt{3}} \sum_{i,j=1}^3 \delta_{ij} \frac{1}{\sqrt{3}} \sum_{k,\ell=1}^3 \delta_{k\ell}}_{\text{meson colour wave functions}} \times \underbrace{\sum_{a=1}^8 \frac{\lambda_{ki}^a}{2} \frac{\lambda_{j\ell}^a}{2}}_{\text{“(colour charge)}^2\text{”}} / g_s^2$$

$$\begin{aligned}
&= \frac{1}{12} \sum_{i,k=1}^3 \sum_{a=1}^8 \lambda_{ki}^a \lambda_{ik}^a \\
&= \frac{1}{12} \underbrace{\sum_{a=1}^8 \text{Tr} [(\lambda^a)^2]}_{16} \\
&= \frac{4}{3} .
\end{aligned}$$

Hence, the one-gluon exchange contribution t_{exch} to the scattering amplitude t is given by the expression

$$t_{\text{exch}} = -\frac{4}{3} \frac{g_s^2}{k^2} \bar{u}(q_1, \tau_1) \gamma_\mu u(p_1, \sigma_1) \bar{v}(p_2, \sigma_2) \gamma^\mu v(q_2, \tau_2) ,$$

which reduces in the nonrelativistic limit to

$$t_{\text{exch}} = \frac{4}{3} \frac{g_s^2}{\mathbf{k}^2} \delta_{\tau_1 \sigma_1} \delta_{\tau_2 \sigma_2} .$$

- The contribution of the pair annihilation graph in Fig. 1.4 to our scattering amplitude t reads

$$t_{\text{ann}} = \frac{g_s^2}{(p_1 + p_2)^2} \frac{\lambda_{ji}^a}{2} \frac{\lambda_{k\ell}^a}{2} \bar{u}(q_1, \tau_1) \gamma_\mu v(q_2, \tau_2) \bar{v}(p_2, \sigma_2) \gamma^\mu u(p_1, \sigma_1) ,$$

or, after the announced multiplication by those meson colour wave functions,

$$\begin{aligned}
t_{\text{ann}} &= \frac{g_s^2}{(p_1 + p_2)^2} \frac{1}{\sqrt{3}} \delta_{ij} \frac{1}{\sqrt{3}} \delta_{k\ell} \frac{\lambda_{ji}^a}{2} \frac{\lambda_{k\ell}^a}{2} \\
&\times \bar{u}(q_1, \tau_1) \gamma_\mu v(q_2, \tau_2) \bar{v}(p_2, \sigma_2) \gamma^\mu u(p_1, \sigma_1) .
\end{aligned}$$

However, this annihilation contribution t_{ann} vanishes identically:

$$\begin{aligned}
\frac{1}{\sqrt{3}} \sum_{i,j=1}^3 \delta_{ij} \frac{\lambda_{ji}^a}{2} &\equiv \frac{1}{2\sqrt{3}} \text{Tr}(\lambda^a) \\
&= 0 .
\end{aligned}$$

From the physical point of view, the interpretation of this, at first sight slightly surprising, phenomenon is rather evident: the gluon, as a colour octet, that is, as a particle which transforms according to the eight-dimensional adjoint representation of $\text{SU}(3)_C$, has no means to couple to a colour singlet, like any bound state of quarks, without violating thereby the conservation of colour demanded by the exact invariance of quantum chromodynamics with respect to the colour gauge group $\text{SU}(3)_C$.

Collecting all our above findings, we may state that, in lowest order of the perturbative loop expansion, the T-matrix element T_{fi} for elastic quark–antiquark scattering within mesons receives only a contribution from the one-gluon exchange graph:

$$T_{\text{fi}} = \frac{1}{(2\pi)^6} t = \frac{1}{(2\pi)^6} \frac{4}{3} \frac{g_s^2}{\mathbf{k}^2} \delta_{\tau_1 \sigma_1} \delta_{\tau_2 \sigma_2} .$$

Consequently, the quintessence of the present consideration is: we may recover the (perturbatively accessible part of the) effective interaction potential operative in quantum chromodynamics from its counterpart in the case of quantum electrodynamics by simply replacing the square e^2 of the electric charge e by the factor

$$\frac{4}{3} g_s^2 .$$

Hence, the short-distance part of the static quark–antiquark potential, arising from one-gluon exchange within mesons, is of Coulombic shape:

$$V_{\text{NR}}^{\text{QCD}}(r) = -\frac{4}{3} \frac{g_s^2}{4\pi r} ,$$

or, with the usually employed definition

$$\alpha_s \equiv \frac{g_s^2}{4\pi}$$

of the strong fine structure constant,

$$\boxed{V_{\text{NR}}^{\text{QCD}}(r) = -\frac{4}{3} \frac{\alpha_s}{r}} .$$

1.4 Lorentz structure of an interquark interaction

At this stage, in order to seize hold of the nonperturbative contribution to any effective potential, we embark on a rather general investigation.

Quite generally, the T-matrix element T_{fi} for the elastic scattering of some generic fermion \mathcal{F} and the corresponding antifermion $\bar{\mathcal{F}}$, both of them of mass m ,

$$\mathcal{F}(p_1, \sigma_1) + \bar{\mathcal{F}}(p_2, \sigma_2) \rightarrow \mathcal{F}(q_1, \tau_1) + \bar{\mathcal{F}}(q_2, \tau_2) ,$$

is, apart from the overall normalization factors of the one-particle wave functions, which we always split off by the definition

$$T_{\text{fi}} =: \frac{1}{(2\pi)^6} \frac{m^2}{\sqrt{E_{p_1} E_{p_2} E_{q_1} E_{q_2}}} t$$

of our scattering amplitude t , the product of

- two bilinears of Dirac spinors of the form $\bar{u}(q_1, \tau_1) \Gamma_1 u(p_1, \sigma_1)$ and $\bar{v}(p_2, \sigma_2) \Gamma_2 v(q_2, \tau_2)$, where Γ_1 and Γ_2 represent some (unspecified) Dirac matrices, and
- some interaction kernel K which, a priori, may depend on all four external momenta p_1, p_2, q_1, q_2 ,

$$K = K(p_1, p_2, q_1, q_2) ,$$

only subject to the momentum conservation

$$p_1 + p_2 = q_1 + q_2 ,$$

as expressed by that overall δ function multiplying this T-matrix element T_{fi} in the standard decomposition of the S-matrix element S_{fi} .

Consequently, the most general ansatz for our scattering amplitude t reads

$$t = \bar{u}(q_1, \tau_1) \Gamma_1 u(p_1, \sigma_1) \bar{v}(p_2, \sigma_2) \Gamma_2 v(q_2, \tau_2) K ,$$

with K depending on any three independent linear combinations built from the external momenta out of the set $\{p_1, p_2, q_1, q_2\}$.

We shall constrain the T-matrix element T_{fi} under consideration by the following two, very reasonable assumptions:

1. The T-matrix element T_{fi} is invariant with respect to the full set of transformations forming the (homogeneous) Lorentz group, that is, invariant under
 - proper orthochronous Lorentz transformations,
 - space reflection (“parity operation”), and
 - time reversal.
2. The interaction kernel K entering in the T-matrix element T_{fi} is a function of only the square k^2 of the involved momentum transfer

$$k \equiv p_1 - q_1 = q_2 - p_2 ;$$

that is,

$$K = K(k^2) ,$$

which reduces in the nonrelativistic limit to

$$K = K(-\mathbf{k}^2) .$$

The most general form of the scattering amplitude t consistent with the requirements of the above assumptions is (see, e.g., Refs. [2, 6, 7])

$$t = \sum_{\Sigma=S,P,V,A,T} t_{\Sigma} , \quad (1.2)$$

where any particular contribution t_{Σ} is of the form

$$t_{\Sigma} = \bar{u}(q_1, \tau_1) \Gamma_{\Sigma} u(p_1, \sigma_1) \bar{v}(p_2, \sigma_2) \Gamma_{\Sigma} v(q_2, \tau_2) K_{\Sigma}(k^2)$$

and the sum extends over the five possible Lorentz structures

- scalar (S),

$$\Gamma_{\Sigma} \otimes \Gamma_{\Sigma} = 1 \otimes 1 ,$$

- pseudoscalar (P),

$$\Gamma_{\Sigma} \otimes \Gamma_{\Sigma} = \gamma_5 \otimes \gamma^5 ,$$

- vector (V),

$$\Gamma_{\Sigma} \otimes \Gamma_{\Sigma} = \gamma_{\mu} \otimes \gamma^{\mu} ,$$

- axial vector (A),

$$\Gamma_{\Sigma} \otimes \Gamma_{\Sigma} = \gamma_{\mu} \gamma_5 \otimes \gamma^{\mu} \gamma^5 ,$$

and

- tensor (T),

$$\Gamma_{\Sigma} \otimes \Gamma_{\Sigma} = \frac{1}{2} \sigma_{\mu\nu} \otimes \sigma^{\mu\nu} .$$

Here, we had to introduce the Dirac matrices

$$\begin{aligned} \gamma_5 \equiv \gamma^5 &:= -\frac{i}{4!} \epsilon_{\mu\nu\rho\sigma} \gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\sigma} \\ &= -i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \\ &= i \gamma_0 \gamma_1 \gamma_2 \gamma_3 , \end{aligned}$$

with the totally antisymmetric Levi–Civita symbol in four dimensions

$$\epsilon_{\mu\nu\rho\sigma} = -\epsilon^{\mu\nu\rho\sigma}$$

unambiguously fixed by demanding

$$\epsilon_{0123} = 1 ,$$

as well as

$$\sigma_{\mu\nu} := \frac{i}{2} [\gamma_{\mu}, \gamma_{\nu}] .$$

In the Dirac representation, these Dirac matrices are explicitly given by

$$\gamma_5 = - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$\begin{aligned} \sigma^{0i} &= i \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad i = 1, 2, 3, \\ \sigma^{ij} &= \epsilon_{ijk} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}, \quad i, j, k = 1, 2, 3. \end{aligned}$$

In the two preceding sections, that fermion–antifermion interaction was basically mediated by the exchange of some vector boson, namely,

- the photon in the case of quantum electrodynamics or
- the gluon in the case of quantum chromodynamics.

As a consequence of this, in both of these cases the effective interaction was solely of vector Lorentz structure,

$$\Gamma_1 \otimes \Gamma_2 = \gamma_\mu \otimes \gamma^\mu,$$

or, in other words, in the decomposition (1.2) the scattering amplitude t received exclusively a vector contribution. The interaction kernel K was given by

$$K(k^2) = -\frac{\kappa}{k^2}$$

reducing to

$$K(-\mathbf{k}^2) = \frac{\kappa}{\mathbf{k}^2},$$

where the effective coupling strength κ stands

- in the case of quantum electrodynamics for

$$\kappa = e^2,$$

- in the case of quantum chromodynamics for

$$\kappa = \frac{4}{3} g_s^2.$$

Fourier transformation then resulted in the static Coulomb potential

$$V_{\text{NR}}(r) = -\frac{\kappa}{4\pi r}.$$

Needless to say, in general the various contributions entering in the decomposition (1.2) of our scattering amplitude t will not arise from the exchange of a single particle representing some fundamental degree of freedom of the underlying quantum field theory. Rather, these terms have to be interpreted as due to only an *effective* exchange of a particle of the appropriate behaviour under Lorentz transformations.

We should be prepared to the fact that—in the course of evaluating below the various terms t_Σ contributing to the T-matrix element T_{fi} —we shall encounter expectation values of the Pauli matrices of the form $\chi_{\tau_1}^\dagger \boldsymbol{\sigma} \chi_{\sigma_1}$ and $\chi_{\sigma_2}^c \dagger \boldsymbol{\sigma} \chi_{\tau_2}^c$. We shall cast the second of these expressions, which involves two-component spinors χ_σ^c representing the spin degrees of freedom of antifermions, defined by

$$\chi_\sigma^c \equiv -i \sigma_2 \chi_\sigma^* ,$$

with the help of the identity

$$\begin{aligned} \sigma_2 \boldsymbol{\sigma} \sigma_2 &= -\boldsymbol{\sigma}^T \\ &= -\boldsymbol{\sigma}^* \end{aligned}$$

into an equivalent form which involves only the two-component spinors χ_σ pertaining to fermions:

$$\begin{aligned} \chi_\sigma^c \dagger \boldsymbol{\sigma} \chi_\tau^c &= (\chi_\sigma^c \dagger \boldsymbol{\sigma} \chi_\tau^c)^T \\ &= (\chi_\sigma^T \sigma_2 \boldsymbol{\sigma} \sigma_2 \chi_\tau^*)^T \\ &= -(\chi_\sigma^T \boldsymbol{\sigma}^T \chi_\tau^*)^T \\ &= -\chi_\tau^\dagger \boldsymbol{\sigma} \chi_\sigma . \end{aligned}$$

We shall find it convenient to abbreviate the expectation values of the Pauli matrices $\boldsymbol{\sigma}$ by introducing the shorthand notation

$$\begin{aligned} \boldsymbol{\sigma}_1 &\equiv \chi_{\tau_1}^\dagger \boldsymbol{\sigma} \chi_{\sigma_1} , \\ \boldsymbol{\sigma}_2 &\equiv -\chi_{\sigma_2}^c \dagger \boldsymbol{\sigma} \chi_{\tau_2}^c \\ &= \chi_{\tau_2}^\dagger \boldsymbol{\sigma} \chi_{\sigma_2} . \end{aligned} \tag{1.3}$$

Now, what we really have to do when trying to follow the steps given in our prescription for the derivation of that (perturbatively accessible part of an) effective interaction potential from the underlying quantum field theory by considering the relevant elastic-scattering problem may be phrased in the following way:

1. Calculate the expectation values of the considered S operator, S_{fi} , or T operator, T_{fi} , with respect to those Fock-space states usually employed in quantum field theory.
2. Look upon these matrix elements as the expectation values of that interaction potential you are searching for, $V(\mathbf{x})$, with respect to the quantum-theoretical bound states and extract this interaction potential by “factorizing off” all remnants of these bound states.

In this and only this (!) sense one may adhere, when switching from the scattering amplitude to the interaction potential, to the identifications of the spin operators \mathbf{S}_1 and \mathbf{S}_2 of fermion and antifermion with half of the expectation values $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ of the Pauli matrices $\boldsymbol{\sigma}$, respectively:

$$\begin{aligned}\mathbf{S}_1 & \text{ “=” } \frac{\boldsymbol{\sigma}_1}{2} , \\ \mathbf{S}_2 & \text{ “=” } \frac{\boldsymbol{\sigma}_2}{2} ,\end{aligned}$$

and therefore

$$\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \text{ “=” } 4 \mathbf{S}_1 \cdot \mathbf{S}_2 .$$

Adopting an admittedly rather symbolical notation, we suppress in the following any reference to both the momenta and the spin degrees of freedom of the involved particles. Nevertheless, at every moment it should be clear what’s going on. With the above at one’s disposal, the nonrelativistic potentials $V_{\text{NR}}^{(\Sigma)}(\mathbf{x})$, $\Sigma = \text{S, P, V, A, T}$, are easily found:

Scalar: For the scalar Lorentz structure, i.e.,

$$\Gamma_{\Sigma} \otimes \Gamma_{\Sigma} = 1 \otimes 1 ,$$

we find for our scattering amplitude t in the nonrelativistic limit

$$\begin{aligned}t_{\text{S}} & \equiv \bar{u} u \bar{v} v K_{\text{S}}(k^2) \\ & = -K_{\text{S}}(-\mathbf{k}^2) .\end{aligned}$$

Upon Fourier transformation, the corresponding static interaction potential $V_{\text{NR}}^{(\text{S})}(\mathbf{x})$ reads

$$V_{\text{NR}}^{(\text{S})}(\mathbf{x}) = V_{\text{S}}(r) ,$$

where $V_{\text{S}}(r)$ is defined by

$$V_{\text{S}}(r) \equiv \frac{1}{(2\pi)^3} \int d^3k \exp(-i \mathbf{k} \cdot \mathbf{x}) K_{\text{S}}(-\mathbf{k}^2) .$$

Accordingly, the scalar Lorentz structure yields a “pure potential” term.

Pseudoscalar: For the pseudoscalar Lorentz structure, i.e.,

$$\Gamma_\Sigma \otimes \Gamma_\Sigma = \gamma_5 \otimes \gamma^5 ,$$

we find for our scattering amplitude t in the nonrelativistic limit

$$\begin{aligned} t_P &\equiv \bar{u} \gamma_5 u \bar{v} \gamma^5 v K_P(k^2) \\ &= u^\dagger \gamma_0 \gamma_5 u v^\dagger \gamma_0 \gamma^5 v K_P(k^2) \\ &= (\chi^\dagger, 0) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \chi \\ 0 \end{pmatrix} (0, \chi^{c\dagger}) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \chi^c \end{pmatrix} K_P(-\mathbf{k}^2) \\ &= 0 . \end{aligned}$$

Consequently, in the nonrelativistic limit, the contribution of this pseudoscalar Lorentz structure vanishes:

$$V_{\text{NR}}^{(\text{P})}(\mathbf{x}) = 0 .$$

Vector: For the vector Lorentz structure, i.e.,

$$\Gamma_\Sigma \otimes \Gamma_\Sigma = \gamma_\mu \otimes \gamma^\mu ,$$

the nonrelativistic limit of our scattering amplitude t has, in fact, already been calculated in Section 1.2:

$$\begin{aligned} t_V &\equiv \bar{u} \gamma_\mu u \bar{v} \gamma^\mu v K_V(k^2) \\ &= K_V(-\mathbf{k}^2) . \end{aligned}$$

Upon Fourier transformation, the corresponding static interaction potential $V_{\text{NR}}^{(\text{V})}(\mathbf{x})$ reads

$$V_{\text{NR}}^{(\text{V})}(\mathbf{x}) = V_V(r) ,$$

where $V_V(r)$ is defined by

$$V_V(r) \equiv -\frac{1}{(2\pi)^3} \int d^3k \exp(-i \mathbf{k} \cdot \mathbf{x}) K_V(-\mathbf{k}^2) .$$

Accordingly, the vector Lorentz structure yields, very similarly to its scalar counterpart, a “pure potential” term.

Axial vector: For the axial vector Lorentz structure, i.e.,

$$\Gamma_\Sigma \otimes \Gamma_\Sigma = \gamma_\mu \gamma_5 \otimes \gamma^\mu \gamma^5 ,$$

we find

- for $\mu = 0$

$$\begin{aligned}
\bar{u} \gamma_0 \gamma_5 u \bar{v} \gamma^0 \gamma^5 v &\equiv u^\dagger \gamma_5 u v^\dagger \gamma^5 v \\
&= (\chi^\dagger, 0) \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \chi \\ 0 \end{pmatrix} \\
&\times (0, \chi^{c\dagger}) \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \chi^c \end{pmatrix} \\
&= 0
\end{aligned}$$

and

- for $\mu = i$

$$\begin{aligned}
\bar{u} \boldsymbol{\gamma} \gamma_5 u \cdot \bar{v} \boldsymbol{\gamma} \gamma^5 v &\equiv u^\dagger \gamma_0 \boldsymbol{\gamma} \gamma_5 u \cdot v^\dagger \gamma^0 \boldsymbol{\gamma} \gamma^5 v \\
&= (\chi^\dagger, 0) \begin{pmatrix} -\boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix} \begin{pmatrix} \chi \\ 0 \end{pmatrix} \\
&\cdot (0, \chi^{c\dagger}) \begin{pmatrix} -\boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix} \begin{pmatrix} 0 \\ \chi^c \end{pmatrix} \\
&= \chi^\dagger \boldsymbol{\sigma} \chi \cdot \chi^{c\dagger} \boldsymbol{\sigma} \chi^c \\
&\equiv -\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 ,
\end{aligned}$$

and therefore for our scattering amplitude t in the nonrelativistic limit

$$\begin{aligned}
t_A &\equiv \bar{u} \gamma_\mu \gamma_5 u \bar{v} \gamma^\mu \gamma^5 v K_A(k^2) \\
&= [\bar{u} \gamma_0 \gamma_5 u \bar{v} \gamma^0 \gamma^5 v - \bar{u} \boldsymbol{\gamma} \gamma_5 u \cdot \bar{v} \boldsymbol{\gamma} \gamma^5 v] K_A(k^2) \\
&= \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 K_A(-\mathbf{k}^2) .
\end{aligned}$$

Upon Fourier transformation, the corresponding static interaction potential $V_{\text{NR}}^{(\text{A})}(\mathbf{x})$ reads

$$V_{\text{NR}}^{(\text{A})}(\mathbf{x}) = 4 \mathbf{S}_1 \cdot \mathbf{S}_2 V_A(r) ,$$

where $V_A(r)$ is defined by

$$V_A(r) \equiv -\frac{1}{(2\pi)^3} \int d^3k \exp(-i \mathbf{k} \cdot \mathbf{x}) K_A(-\mathbf{k}^2) .$$

Accordingly, the axial vector Lorentz structure entails an effective spin–spin interaction.

Tensor: For the tensor Lorentz structure, i.e.,

$$\Gamma_\Sigma \otimes \Gamma_\Sigma = \frac{1}{2} \sigma_{\mu\nu} \otimes \sigma^{\mu\nu} ,$$

we find

- for $\mu = 0, \nu = i$

$$\begin{aligned} \bar{u} \sigma^{0i} u \bar{v} \sigma^{0i} v &\equiv u^\dagger \gamma_0 \sigma^{0i} u v^\dagger \gamma^0 \sigma^{0i} v \\ &= i (\chi^\dagger, 0) \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \begin{pmatrix} \chi \\ 0 \end{pmatrix} \\ &\times i (0, \chi^{c\dagger}) \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \chi^c \end{pmatrix} \\ &= 0 \end{aligned}$$

and

- for $\mu = i, \nu = j$

$$\begin{aligned} \bar{u} \sigma^{ij} u \bar{v} \sigma^{ij} v &\equiv u^\dagger \gamma_0 \sigma^{ij} u v^\dagger \gamma^0 \sigma^{ij} v \\ &= \epsilon_{ijk} (\chi^\dagger, 0) \begin{pmatrix} \sigma_k & 0 \\ 0 & -\sigma_k \end{pmatrix} \begin{pmatrix} \chi \\ 0 \end{pmatrix} \\ &\times \epsilon_{ijl} (0, \chi^{c\dagger}) \begin{pmatrix} \sigma_l & 0 \\ 0 & -\sigma_l \end{pmatrix} \begin{pmatrix} 0 \\ \chi^c \end{pmatrix} \\ &= -2 \chi^\dagger \sigma_k \chi \chi^{c\dagger} \sigma_k \chi^c \\ &\equiv -2 \chi^\dagger \boldsymbol{\sigma} \chi \cdot \chi^{c\dagger} \boldsymbol{\sigma} \chi^c \\ &\equiv 2 \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 , \end{aligned}$$

and therefore for our scattering amplitude t in the nonrelativistic limit

$$\begin{aligned} t_T &\equiv \frac{1}{2} \bar{u} \sigma_{\mu\nu} u \bar{v} \sigma^{\mu\nu} v K_T(k^2) \\ &= \left[\bar{u} \sigma_{0i} u \bar{v} \sigma^{0i} v + \frac{1}{2} \bar{u} \sigma_{ij} u \bar{v} \sigma^{ij} v \right] K_T(k^2) \\ &= \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 K_T(-\mathbf{k}^2) . \end{aligned}$$

Upon Fourier transformation, the corresponding static interaction potential $V_{\text{NR}}^{(\text{T})}(\mathbf{x})$ reads

$$V_{\text{NR}}^{(\text{T})}(\mathbf{x}) = 4 \mathbf{S}_1 \cdot \mathbf{S}_2 V_T(r) ,$$

where $V_T(r)$ is defined by

$$V_T(r) \equiv -\frac{1}{(2\pi)^3} \int d^3k \exp(-i \mathbf{k} \cdot \mathbf{x}) K_T(-\mathbf{k}^2) .$$

Accordingly, the tensor Lorentz structure entails also an effective spin–spin interaction.

Table 1.1 summarizes our findings for the contributions of the various possible Lorentz structures to the effective interaction potential in the nonrelativistic limit.

Table 1.1: Nonrelativistic interaction potential $V_{\text{NR}}^{(\Sigma)}$ arising effectively from the various conceivable Lorentz structures $\Gamma_\Sigma \otimes \Gamma_\Sigma$ of an arbitrary fermion–antifermion interaction

Lorentz structure	$\Gamma_\Sigma \otimes \Gamma_\Sigma$	static potential
scalar	$1 \otimes 1$	$V_S(r)$
pseudoscalar	$\gamma_5 \otimes \gamma^5$	0
vector	$\gamma_\mu \otimes \gamma^\mu$	$V_V(r)$
axial vector	$\gamma_\mu \gamma_5 \otimes \gamma^\mu \gamma^5$	$4 \mathbf{S}_1 \cdot \mathbf{S}_2 V_A(r)$
tensor	$\frac{1}{2} \sigma_{\mu\nu} \otimes \sigma^{\mu\nu}$	$4 \mathbf{S}_1 \cdot \mathbf{S}_2 V_T(r)$

The total spin \mathbf{S} of the respective bound state under consideration is clearly given by the sum of the spins \mathbf{S}_1 and \mathbf{S}_2 of its constituents:

$$\mathbf{S} \equiv \mathbf{S}_1 + \mathbf{S}_2 .$$

Upon squaring this relation,

$$\begin{aligned} \mathbf{S}^2 &= (\mathbf{S}_1 + \mathbf{S}_2)^2 \\ &= \mathbf{S}_1^2 + \mathbf{S}_2^2 + 2 \mathbf{S}_1 \cdot \mathbf{S}_2 , \end{aligned}$$

we may express the product $\mathbf{S}_1 \cdot \mathbf{S}_2$ of the two spins \mathbf{S}_1 and \mathbf{S}_2 in terms of the squares of \mathbf{S}_1 , \mathbf{S}_2 , and \mathbf{S} ,

$$\mathbf{S}_1 \cdot \mathbf{S}_2 = \frac{1}{2} (\mathbf{S}^2 - \mathbf{S}_1^2 - \mathbf{S}_2^2) ,$$

and, therefore, its expectation values by the corresponding expectation values of \mathbf{S}_1^2 , \mathbf{S}_2^2 , and \mathbf{S}^2 :

$$\langle \mathbf{S}_1 \cdot \mathbf{S}_2 \rangle = \frac{1}{2} (\langle \mathbf{S}^2 \rangle - \langle \mathbf{S}_1^2 \rangle - \langle \mathbf{S}_2^2 \rangle) .$$

Accordingly, expressed in terms of the quantum numbers S , S_1 , and S_2 of the spins \mathbf{S} , \mathbf{S}_1 , and \mathbf{S}_2 , respectively, the expectation values $\langle \mathbf{S}_1 \cdot \mathbf{S}_2 \rangle$ of the product of the spins \mathbf{S}_1 , \mathbf{S}_2 of the bound-state constituents read

$$\langle \mathbf{S}_1 \cdot \mathbf{S}_2 \rangle = \frac{1}{2} [S(S+1) - S_1(S_1+1) - S_2(S_2+1)] .$$

For fermionic constituents with spin

$$S_1 = S_2 = \frac{1}{2} ,$$

we have

$$S_1(S_1+1) = S_2(S_2+1) = \frac{3}{4}$$

and therefore

$$\langle \mathbf{S}_1 \cdot \mathbf{S}_2 \rangle = \frac{1}{2} S(S+1) - \frac{3}{4} .$$

Moreover, for fermionic constituents with spin

$$S_1 = S_2 = \frac{1}{2} ,$$

the quantum number S of the total spin \mathbf{S} may accept precisely either of two values:

- $S = 0$, which corresponds to some spin singlet, like the pion or the η meson in the case of light quarks, or the η_c in the charmonium system.
- $S = 1$, which corresponds to some spin triplet, like the ρ , ω , and ϕ mesons in the case of light quarks, or the J/ψ in the charmonium system, or the Υ in the bottomonium system.

This implies for the eigenvalues $S(S+1)$ of the square \mathbf{S}^2 of the total spin \mathbf{S} :

$$S(S+1) = \begin{cases} 0 & \text{for spin singlets, i.e., } S = 0 , \\ 2 & \text{for spin triplets, i.e., } S = 1 . \end{cases}$$

Accordingly, the expectation values $\langle \mathbf{S}_1 \cdot \mathbf{S}_2 \rangle$ of the product of the spins \mathbf{S}_1 , \mathbf{S}_2 of the bound-state constituents are finally given by

$$\langle \mathbf{S}_1 \cdot \mathbf{S}_2 \rangle = \begin{cases} -\frac{3}{4} & \text{for spin singlets, i.e., } S = 0 , \\ +\frac{1}{4} & \text{for spin triplets, i.e., } S = 1 . \end{cases} \quad (1.4)$$

The first and simultaneously most important lesson to be learned from the above is that, for these two possible values of the quantum number S of the total spin \mathbf{S} of the bound state, the spin–spin interaction term will contribute necessarily with opposite signs.

Collecting all previous results, the following picture emerges for the *nonrelativistic limit* of the most general effective fermion–antifermion interaction potential $V_{\text{NR}}(\mathbf{x})$:

- Both scalar and vector Lorentz structures lead to “pure potential” terms:

$$V_{\text{NR}}^{(\Sigma)}(\mathbf{x}) = V_{\Sigma}(r) \quad \text{for} \quad \Sigma = S, V .$$

- The contribution of the pseudoscalar Lorentz structure vanishes:¹

$$V_{\text{NR}}^{(\text{P})}(\mathbf{x}) = 0 .$$

This circumstance provides, for instance, a very compelling reason for the (relatively) weak binding of deuterium in nuclear physics: The interaction between nucleons, that is, protons and neutrons, is generally accepted to be dominated by one-pion exchange. Since the π meson is a *pseudoscalar* meson, the Lorentz structure of its coupling to the nucleons has to be also of pseudoscalar nature in order to form an interaction Lagrangian which is a Lorentz scalar. This fact implies that only the relativistic corrections arising from one-pion exchange can be responsible for the binding of a proton and a neutron to the deuterium.

- Both axial vector and tensor Lorentz structures contribute merely to the spin–spin interaction term:

$$V_{\text{NR}}^{(\Sigma)}(\mathbf{x}) = 4 \mathbf{S}_1 \cdot \mathbf{S}_2 V_{\Sigma}(r) \quad \text{for} \quad \Sigma = A, T .$$

As a consequence of Eq. (1.4), in this case we will obtain a binding force between fermion and antifermion only for either of the above two possible values, $S = 0$ and $S = 1$, of the quantum number S of the total spin \mathbf{S} of the bound state.

Hoping that the empirically observed hadron spectrum will provide some restrictions on the allowed effective quark–antiquark interaction, we now confront the above picture of general findings with experiment:

¹ In the next—i.e., first non-trivial—order of the present nonrelativistic expansion, this pseudoscalar Lorentz structure contributes to the spin–spin and tensor interaction terms.

- Already the mere existence of strongly bound mesons forbids the pseudoscalar Lorentz structure to play any significant rôle within some phenomenologically acceptable quark–antiquark interaction potential.
- The existence of both pseudoscalar mesons (like π , η , and η') *and* vector mesons (like ρ , ω , and ϕ), all of which are bound states of a quark–antiquark pair with vanishing orbital angular momentum, implies that the actual quark–antiquark forces must be described by an interaction potential which yields binding for $S = 0$ as well as $S = 1$. Obviously, this fact rules out both the axial vector and tensor Lorentz structures as the predominant contribution to any realistic quark–antiquark interaction potential.

In other words, the theoretically predicted particle spectra would look very different from the experimentally measured ones if the dominant terms in the effective quark–antiquark interaction potential would not be just some linear combination of vector and scalar Lorentz structure. Therefore, our conclusion has to be:

The Lorentz structure of the quark–antiquark interaction is dominated by the vector $\gamma_\mu \otimes \gamma^\mu$ and/or the scalar $1 \otimes 1$, both of which lead in the nonrelativistic limit to so-called pure potential terms. Thus the static interaction potential $V_{\text{NR}}(r)$ must be the sum of merely the contributions of the vector— $V_V(r)$ —and the scalar— $V_S(r)$ —Lorentz structures:

$$V_{\text{NR}}(r) = V_V(r) + V_S(r) .$$

Chapter 2

Relativistic Corrections

Beyond doubt, the next logical step must be to improve the up-to-now entirely nonrelativistic formalism by taking into account all relativistic corrections. In principle, one encounters no particular difficulties when trying to take into account (at least, at some formal level) the complete set of relativistic corrections to the effective interaction potential [4, 5].

For the moment, however, we intend to be somewhat more modest, and this even in two respects:

1. We shall calculate these relativistic corrections only up to second order in the absolute value $v \equiv |\mathbf{v}|$ of the generic relative velocities

$$\mathbf{v} = \frac{\mathbf{p}}{E_p}$$

of the bound-state constituents, that is, only up to order

$$v^2 = \frac{\mathbf{p}^2}{E_p^2} ,$$

which, since up to this order the relativistic kinetic energy

$$E_p = \sqrt{\mathbf{p}^2 + m^2}$$

may be approximated at this place by

$$E_p \simeq m ,$$

is equivalent to

$$v^2 \simeq \frac{\mathbf{p}^2}{m^2} .$$

2. We shall consider only the *spin-dependent* contributions to these relativistic corrections. These spin-dependent interactions control the fine and hyperfine level splittings of the bound-state spectra. The *spin-independent* interactions may be obtained, with slightly more effort, along similar lines [6].

2.1 Spin-dependent corrections

We shall be interested in all the spin-dependent relativistic corrections to the static interaction potential $V_{\text{NR}}(\mathbf{x})$ up to order

$$\frac{\mathbf{p}^2}{m^2} .$$

Therefore, we must focus our attention to those terms in the transition amplitude T_{fi} which involve expectation values of the Pauli matrices $\boldsymbol{\sigma}$. One may simplify this task considerably by the following observation: we are entitled to approximate the relativistic kinetic energy

$$E_p = \sqrt{\mathbf{p}^2 + m^2}$$

by the lowest-order term

$$E_p \simeq m$$

of its nonrelativistic expansion in two places, namely,

1. in the Dirac spinors $u(p, \sigma)$ and $v(p, \sigma)$,

$$\begin{aligned} u(p, \sigma) &= \sqrt{\frac{E_p + m}{2m}} \begin{pmatrix} 1 \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E_p + m} \end{pmatrix} \chi_\sigma , \\ v(p, \sigma) &= \sqrt{\frac{E_p + m}{2m}} \begin{pmatrix} \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E_p + m} \\ 1 \end{pmatrix} \chi_\sigma^c , \quad \chi_\sigma^c \equiv -i \sigma_2 \chi_\sigma^* , \end{aligned}$$

which then (only for the purpose of the present analysis!) become

$$\begin{aligned} u(p, \sigma) &= \begin{pmatrix} 1 \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{2m} \end{pmatrix} \chi_\sigma , \\ v(p, \sigma) &= \begin{pmatrix} \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{2m} \\ 1 \end{pmatrix} \chi_\sigma^c ; \end{aligned} \tag{2.1}$$

2. in that general relationship between the T-matrix element T_{fi} and our scattering amplitude t ,

$$T_{\text{fi}} =: \frac{1}{(2\pi)^6} \frac{m^2}{\sqrt{E_{p_1} E_{p_2} E_{q_1} E_{q_2}}} t ,$$

which then becomes

$$T_{\text{fi}} = \frac{1}{(2\pi)^6} t .$$

In the course of calculating the scattering amplitude t , we may take advantage of two trivial simplifications:

1. In order to get the interaction potential $V(\mathbf{x})$, we have to consider the scattering amplitude t only in the center-of-momentum frame, defined by the vanishing of the total momenta \mathbf{P}_i and \mathbf{P}_f of initial state i and final state f , respectively:

$$\mathbf{P}_i \equiv \mathbf{p}_1 + \mathbf{p}_2 = \mathbf{q}_1 + \mathbf{q}_2 \equiv \mathbf{P}_f = 0 .$$

Consequently, this scattering amplitude t will depend only on the involved momentum transfer

$$\mathbf{k} \equiv \mathbf{p}_1 - \mathbf{q}_1 = \mathbf{q}_2 - \mathbf{p}_2$$

and on the relative momentum

$$\mathbf{p} \equiv \mathbf{p}_1 = -\mathbf{p}_2 .$$

2. The Pauli matrices $\boldsymbol{\sigma} \equiv \{\sigma_i, i = 1, 2, 3\}$ satisfy both

- the commutation relations

$$[\sigma_i, \sigma_j] = 2i \epsilon_{ijk} \sigma_k$$

and

- the anticommutation relations

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij} .$$

Adding up these two relations, the product $\sigma_i \sigma_j$ of any two Pauli matrices is given by

$$\begin{aligned} \sigma_i \sigma_j &= \frac{1}{2} ([\sigma_i, \sigma_j] + \{\sigma_i, \sigma_j\}) \\ &= \delta_{ij} + i \epsilon_{ijk} \sigma_k . \end{aligned}$$

By application of this relation, any product of (two or more) Pauli matrices may be reduced to an expression which involves no more than at most *one* Pauli matrix.

Performing the Fourier transformation as demanded by that general prescription briefly sketched in our introductory remarks to Chapter 1, the effective interaction potential $V(\mathbf{x})$ is derived from t according to

$$\begin{aligned} V(\mathbf{x}) &= -(2\pi)^3 \int d^3k \exp(-i \mathbf{k} \cdot \mathbf{x}) T_{fi}(k) \\ &= -\frac{1}{(2\pi)^3} \int d^3k \exp(-i \mathbf{k} \cdot \mathbf{x}) t . \end{aligned}$$

As the central result of the intended inclusion of all spin-dependent relativistic corrections up to order

$$\frac{\mathbf{p}^2}{m^2} ,$$

we shall finally end up with the generalization to arbitrary interaction potentials of the well-known *Breit–Fermi Hamiltonian*, of the standard form

$$H = m_1 + m_2 + \frac{\mathbf{p}^2}{2\mu} + V(\mathbf{x}) ,$$

with μ the reduced mass of the particular two-particle quantum system under consideration,

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2} .$$

Here, the interaction potential $V(\mathbf{x})$ will encompass, in addition to the nonrelativistic contribution $V_{\text{NR}}(\mathbf{x})$, also all spin-dependent relativistic corrections $V_{\text{spin}}(\mathbf{x})$:

$$V(\mathbf{x}) = V_{\text{NR}}(\mathbf{x}) + V_{\text{spin}}(\mathbf{x}) .$$

The set of spin-dependent relativistic corrections $V_{\text{spin}}(\mathbf{x})$ will turn out to consist, in general, of some

- spin–orbit interaction term H_{LS} ,
- spin–spin interaction term H_{SS} , and
- tensor interaction term H_{T} ;

that is, this spin-dependent part V_{spin} of the interaction potential $V(\mathbf{x})$ will read

$$V_{\text{spin}} = H_{\text{LS}} + H_{\text{SS}} + H_{\text{T}} .$$

Bearing in mind the outcome of our analysis of the possible Lorentz structure of the effective interaction in a quark–antiquark bound state as performed in Section 1.4, we will treat below only the case of vector and scalar Lorentz structure of the static interaction potential $V_{\text{NR}}(r)$:

$$V_{\text{NR}}(r) = V_{\text{V}}(r) + V_{\text{S}}(r) .$$

Furthermore, for the sake of simplicity, we will present in the following all the necessary derivations in detail only for the special case of equal masses of the bound-state constituents:

$$m_1 = m_2 = m .$$

2.2 Interaction with vector Lorentz structure

In the case of an interaction with vector Lorentz structure, that is, for

$$\Gamma_\Sigma \otimes \Gamma_\Sigma = \gamma_\mu \otimes \gamma^\mu ,$$

the scattering amplitude (1.2) assumes the form

$$t_V \equiv \bar{u}(q_1, \tau_1) \gamma_\mu u(p_1, \sigma_1) \bar{v}(p_2, \sigma_2) \gamma^\mu v(q_2, \tau_2) K_V(k^2) .$$

Upon inserting the Dirac spinors in Dirac representation, Eq. (2.1), and suppressing, for the sake of notational simplicity, any reference to the spin degrees of freedom, we find for this scattering amplitude

$$\begin{aligned} t_V &\equiv \bar{u}(q_1) \gamma_\mu u(p_1) \bar{v}(p_2) \gamma^\mu v(q_2) K_V(k^2) \\ &\equiv [u^\dagger(q_1) u(p_1) v^\dagger(p_2) v(q_2) \\ &\quad - u^\dagger(q_1) \gamma_0 \boldsymbol{\gamma} u(p_1) \cdot v^\dagger(p_2) \gamma_0 \boldsymbol{\gamma} v(q_2)] K_V(k^2) \\ &= \left[\chi^\dagger \left(1, \frac{\boldsymbol{\sigma} \cdot \mathbf{q}_1}{2m} \right) \begin{pmatrix} 1 \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}_1}{2m} \end{pmatrix} \chi \chi^{c\dagger} \left(\frac{\boldsymbol{\sigma} \cdot \mathbf{p}_2}{2m}, 1 \right) \begin{pmatrix} \frac{\boldsymbol{\sigma} \cdot \mathbf{q}_2}{2m} \\ 1 \end{pmatrix} \chi^c \right. \\ &\quad - \chi^\dagger \left(1, \frac{\boldsymbol{\sigma} \cdot \mathbf{q}_1}{2m} \right) \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}_1}{2m} \end{pmatrix} \chi \\ &\quad \cdot \left. \chi^{c\dagger} \left(\frac{\boldsymbol{\sigma} \cdot \mathbf{p}_2}{2m}, 1 \right) \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \begin{pmatrix} \frac{\boldsymbol{\sigma} \cdot \mathbf{q}_2}{2m} \\ 1 \end{pmatrix} \chi^c \right] K_V(-\mathbf{k}^2) \\ &= \left\{ \left[\chi^\dagger \chi + \frac{1}{4m^2} \chi^\dagger (\boldsymbol{\sigma} \cdot \mathbf{q}_1) (\boldsymbol{\sigma} \cdot \mathbf{p}_1) \chi \right] \right. \\ &\quad \times \left[\chi^{c\dagger} \chi^c + \frac{1}{4m^2} \chi^{c\dagger} (\boldsymbol{\sigma} \cdot \mathbf{p}_2) (\boldsymbol{\sigma} \cdot \mathbf{q}_2) \chi^c \right] \\ &\quad - \frac{1}{4m^2} [\chi^\dagger \boldsymbol{\sigma} (\boldsymbol{\sigma} \cdot \mathbf{p}_1) \chi + \chi^\dagger (\boldsymbol{\sigma} \cdot \mathbf{q}_1) \boldsymbol{\sigma} \chi] \\ &\quad \cdot \left. [\chi^{c\dagger} (\boldsymbol{\sigma} \cdot \mathbf{p}_2) \boldsymbol{\sigma} \chi^c + \chi^{c\dagger} \boldsymbol{\sigma} (\boldsymbol{\sigma} \cdot \mathbf{q}_2) \chi^c] \right\} K_V(-\mathbf{k}^2) \\ &= \left\{ 1 + \frac{1}{4m^2} \left[\chi^\dagger (\boldsymbol{\sigma} \cdot \mathbf{q}_1) (\boldsymbol{\sigma} \cdot \mathbf{p}_1) \chi + \chi^{c\dagger} (\boldsymbol{\sigma} \cdot \mathbf{p}_2) (\boldsymbol{\sigma} \cdot \mathbf{q}_2) \chi^c \right] \right. \\ &\quad - \chi^\dagger [\boldsymbol{\sigma} (\boldsymbol{\sigma} \cdot \mathbf{p}_1) + (\boldsymbol{\sigma} \cdot \mathbf{q}_1) \boldsymbol{\sigma}] \chi \\ &\quad \cdot \left. \chi^{c\dagger} [(\boldsymbol{\sigma} \cdot \mathbf{p}_2) \boldsymbol{\sigma} + \boldsymbol{\sigma} (\boldsymbol{\sigma} \cdot \mathbf{q}_2)] \chi^c \right\} K_V(-\mathbf{k}^2) . \end{aligned}$$

Dropping all contributions to spin-independent relativistic corrections at the very instant they show up, and recalling the abbreviations (1.3), this scattering amplitude simplifies, in the center-of-momentum frame, to

$$\begin{aligned}
t_V &= \left\{ 1 + \frac{1}{4m^2} \{ i \epsilon_{ijk} p_i k_j \sigma_{1k} + i \epsilon_{ijk} p_i k_j \sigma_{2k} \right. \\
&\quad \left. + [(2p - k)_i + i \epsilon_{ijk} k_j \sigma_{1k}] [(2p - k)_i + i \epsilon_{ilm} k_\ell \sigma_{2m}] \} \right\} K_V(-\mathbf{k}^2) \\
&= \left\{ 1 + \frac{1}{4m^2} [3 i \epsilon_{ijk} p_i k_j (\sigma_{1k} + \sigma_{2k}) \right. \\
&\quad \left. - (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2) \mathbf{k}^2 + (\boldsymbol{\sigma}_1 \cdot \mathbf{k}) (\boldsymbol{\sigma}_2 \cdot \mathbf{k})] \right\} K_V(-\mathbf{k}^2) .
\end{aligned}$$

Fourier transformation appears to be the appropriate tool to obtain the resulting effective interaction potential:

- Already from the very beginning, all interaction kernels K_Σ have been (implicitly) assumed to depend, in the nonrelativistic limit, only on the modulus $|\mathbf{k}|$ of the momentum transfer \mathbf{k} . Therefore, all the corresponding static interaction potentials $V_{\text{NR}}^{(\Sigma)}(\mathbf{x})$ have to be spherically symmetric:

$$V_V(r) \equiv -\frac{1}{(2\pi)^3} \int d^3k \exp(-i \mathbf{k} \cdot \mathbf{x}) K_V(-\mathbf{k}^2) .$$

- Denoting the first and second derivatives of any static interaction potential with respect to the radial coordinate $r \equiv |\mathbf{x}|$ by primes, one finds

$$\begin{aligned}
\frac{1}{(2\pi)^3} \int d^3k k_j \exp(-i \mathbf{k} \cdot \mathbf{x}) K_V(-\mathbf{k}^2) &= -i \nabla_j V_V(r) \\
&= -i \frac{x_j}{r} V_V'(r) ,
\end{aligned}$$

$$\frac{1}{(2\pi)^3} \int d^3k k^2 \exp(-i \mathbf{k} \cdot \mathbf{x}) K_V(-\mathbf{k}^2) = \Delta V_V(r) ,$$

and, with the help of an identity proven in Appendix E,

$$\begin{aligned}
&\frac{1}{(2\pi)^3} \int d^3k k_i k_j \exp(-i \mathbf{k} \cdot \mathbf{x}) K_V(-\mathbf{k}^2) \\
&= \nabla_i \nabla_j V_V(r) \\
&= \left(\frac{x_i x_j}{r^2} - \frac{1}{3} \delta_{ij} \right) \left[V_V''(r) - \frac{1}{r} V_V'(r) \right] + \frac{1}{3} \delta_{ij} \Delta V_V(r) .
\end{aligned}$$

Consequently, the spin-dependent relativistic corrections $V_{\text{spin}}^{\text{vector}}$ for the case of a vector Lorentz structure of the effective fermion–antifermion interaction become

$$\begin{aligned} V_{\text{spin}}^{\text{vector}} &= \frac{3}{2m^2 r} (\mathbf{x} \times \mathbf{p}) \cdot \mathbf{S} V_V'(r) \\ &+ \frac{2}{3m^2} \mathbf{S}_1 \cdot \mathbf{S}_2 \Delta V_V(r) \\ &+ \frac{1}{m^2} \left[\frac{(\mathbf{S}_1 \cdot \mathbf{x})(\mathbf{S}_2 \cdot \mathbf{x})}{r^2} - \frac{1}{3} \mathbf{S}_1 \cdot \mathbf{S}_2 \right] \left[\frac{1}{r} V_V'(r) - V_V''(r) \right] . \end{aligned}$$

Herein, it is straightforward to identify, in full accordance with the previously announced decomposition of the spin-dependent relativistic corrections V_{spin} , when specified to the case of vector Lorentz structure,

$$V_{\text{spin}}^{\text{vector}} = H_{\text{LS}}^{\text{vector}} + H_{\text{SS}}^{\text{vector}} + H_{\text{T}}^{\text{vector}} ,$$

- the spin–orbit term

$$\begin{aligned} H_{\text{LS}}^{\text{vector}} &= \frac{3}{2m^2 r} (\mathbf{x} \times \mathbf{p}) \cdot \mathbf{S} V_V'(r) \\ &\equiv \frac{3}{2m^2 r} \mathbf{L} \cdot \mathbf{S} V_V'(r) , \end{aligned}$$

with the relative orbital angular momentum

$$\mathbf{L} \equiv \mathbf{x} \times \mathbf{p}$$

of the bound-state constituents;

- the spin–spin term

$$H_{\text{SS}}^{\text{vector}} = \frac{2}{3m^2} \mathbf{S}_1 \cdot \mathbf{S}_2 \Delta V_V(r) ;$$

and

- the tensor term

$$\begin{aligned} H_{\text{T}}^{\text{vector}} &= \frac{1}{m^2} \left[\frac{(\mathbf{S}_1 \cdot \mathbf{x})(\mathbf{S}_2 \cdot \mathbf{x})}{r^2} - \frac{1}{3} \mathbf{S}_1 \cdot \mathbf{S}_2 \right] \left[\frac{1}{r} V_V'(r) - V_V''(r) \right] \\ &= \frac{1}{12m^2} S_{12} \left[\frac{1}{r} V_V'(r) - V_V''(r) \right] , \end{aligned}$$

with the shorthand notation

$$S_{12} \equiv 12 \left[\frac{(\mathbf{S}_1 \cdot \mathbf{x})(\mathbf{S}_2 \cdot \mathbf{x})}{r^2} - \frac{1}{3} \mathbf{S}_1 \cdot \mathbf{S}_2 \right]$$

for the spin-dependent factor.

2.3 Interaction with scalar Lorentz structure

In the case of an interaction with scalar Lorentz structure, that is, for

$$\Gamma_\Sigma \otimes \Gamma_\Sigma = 1 \otimes 1 ,$$

the scattering amplitude (1.2) assumes the form

$$t_S \equiv \bar{u}(q_1, \tau_1) u(p_1, \sigma_1) \bar{v}(p_2, \sigma_2) v(q_2, \tau_2) K_S(k^2) .$$

Upon inserting the Dirac spinors in Dirac representation, Eq. (2.1), and suppressing, for the sake of notational simplicity, any reference to the spin degrees of freedom, we find for this scattering amplitude

$$\begin{aligned} t_S &\equiv \bar{u}(q_1) u(p_1) \bar{v}(p_2) v(q_2) K_S(k^2) \\ &\equiv u^\dagger(q_1) \gamma_0 u(p_1) v^\dagger(p_2) \gamma_0 v(q_2) K_S(k^2) \\ &= \chi^\dagger \left(1, \frac{\boldsymbol{\sigma} \cdot \mathbf{q}_1}{2m} \right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}_1}{2m} \end{pmatrix} \chi \\ &\times \chi^{c\dagger} \left(\frac{\boldsymbol{\sigma} \cdot \mathbf{p}_2}{2m}, 1 \right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{\boldsymbol{\sigma} \cdot \mathbf{q}_2}{2m} \\ 1 \end{pmatrix} \chi^c K_S(-\mathbf{k}^2) \\ &= - \left[\chi^\dagger \chi - \frac{1}{4m^2} \chi^\dagger (\boldsymbol{\sigma} \cdot \mathbf{q}_1) (\boldsymbol{\sigma} \cdot \mathbf{p}_1) \chi \right] \\ &\quad \times \left[\chi^{c\dagger} \chi^c - \frac{1}{4m^2} \chi^{c\dagger} (\boldsymbol{\sigma} \cdot \mathbf{p}_2) (\boldsymbol{\sigma} \cdot \mathbf{q}_2) \chi^c \right] K_S(-\mathbf{k}^2) \\ &= - \left\{ 1 - \frac{1}{4m^2} [\chi^\dagger (\boldsymbol{\sigma} \cdot \mathbf{q}_1) (\boldsymbol{\sigma} \cdot \mathbf{p}_1) \chi + \chi^{c\dagger} (\boldsymbol{\sigma} \cdot \mathbf{p}_2) (\boldsymbol{\sigma} \cdot \mathbf{q}_2) \chi^c] \right\} \\ &\quad \times K_S(-\mathbf{k}^2) . \end{aligned}$$

Dropping all contributions to spin-independent relativistic corrections at the very instant they show up, and recalling the abbreviations (1.3), this scattering amplitude simplifies, in the center-of-momentum frame, to

$$\begin{aligned} t_S &= - \left[1 - \frac{1}{4m^2} (i \epsilon_{ijk} p_i k_j \sigma_{1k} + i \epsilon_{ijk} p_i k_j \sigma_{2k}) \right] K_S(-\mathbf{k}^2) \\ &= - \left[1 - \frac{1}{4m^2} i \epsilon_{ijk} p_i k_j (\sigma_{1k} + \sigma_{2k}) \right] K_S(-\mathbf{k}^2) . \end{aligned}$$

Once more, Fourier transformation appears to be the adequate tool to obtain the resulting effective interaction potential:

- For the same reason as before, the static interaction potential has to be again spherically symmetric:

$$V_S(r) \equiv \frac{1}{(2\pi)^3} \int d^3k \exp(-i \mathbf{k} \cdot \mathbf{x}) K_S(-\mathbf{k}^2) .$$

- The corresponding spin-dependent relativistic corrections may be found with the help of

$$\begin{aligned} \frac{1}{(2\pi)^3} \int d^3k k_j \exp(-i \mathbf{k} \cdot \mathbf{x}) K_S(-\mathbf{k}^2) &= i \nabla_j V_S(r) \\ &= i \frac{x_j}{r} V'_S(r) . \end{aligned}$$

Consequently, the spin-dependent relativistic corrections $V_{\text{spin}}^{\text{scalar}}$ for the case of a scalar Lorentz structure of the effective fermion–antifermion interaction become

$$V_{\text{spin}}^{\text{scalar}} = H_{\text{LS}}^{\text{scalar}} ,$$

with the spin–orbit term

$$H_{\text{LS}}^{\text{scalar}} = -\frac{1}{2m^2 r} \mathbf{L} \cdot \mathbf{S} V'_S(r) ,$$

where \mathbf{L} denotes, as before, the relative orbital angular momentum of the bound-state constituents,

$$\mathbf{L} \equiv \mathbf{x} \times \mathbf{p} .$$

Accordingly, an interaction with scalar spin structure contributes only to the spin–orbit term H_{LS} . However, apart from possible differences of the two nonrelativistic potentials $V_V(r)$ and $V_S(r)$ of vector and scalar spin structure, respectively, the spin-dependent relativistic corrections for scalar spin structure, $H_{\text{LS}}^{\text{scalar}}$, contribute with a sign opposite to that of the corresponding spin-dependent relativistic corrections for vector spin structure, $H_{\text{LS}}^{\text{vector}}$. Hence, assuming identical static potentials, i.e.,

$$V_V(r) = V_S(r) ,$$

the spin–orbit term $H_{\text{LS}}^{\text{vector}}$, resulting from a vector spin structure, may be partially compensated by the spin–orbit term $H_{\text{LS}}^{\text{scalar}}$, resulting from a scalar spin structure:

$$H_{\text{LS}}^{\text{scalar}} = -\frac{1}{3} H_{\text{LS}}^{\text{vector}} \quad \text{for} \quad V_V(r) = V_S(r) .$$

Before trying, in Chapter 3, to write down a (physically meaningful) quark–antiquark interaction potential, it is advisable to “condense” all these results on the effective fermion–antifermion interaction potential to what we would like to call a “generalized Breit–Fermi Hamiltonian.”

2.4 Generalized Breit–Fermi Hamiltonian

The Hamiltonian containing all spin-dependent relativistic corrections up to order

$$v^2 = \frac{\mathbf{p}^2}{E_p^2} \simeq \frac{\mathbf{p}^2}{m^2}$$

is called the *generalized Breit–Fermi Hamiltonian*:

$$\boxed{H = m_1 + m_2 + \frac{\mathbf{p}^2}{2\mu} + V_{\text{NR}}(r) + H_{\text{LS}} + H_{\text{SS}} + H_{\text{T}}}, \quad (2.2)$$

where

$$\begin{aligned} \mu &\equiv \frac{m_1 m_2}{m_1 + m_2} \\ &= \frac{m}{2} \quad \text{for } m_1 = m_2 = m \end{aligned}$$

is the reduced mass and—according to the analysis of Section 1.4—the static potential $V_{\text{NR}}(r)$ consists of a vector and a scalar contribution,

$$\boxed{V_{\text{NR}}(r) = V_{\text{V}}(r) + V_{\text{S}}(r)}. \quad (2.3)$$

The corresponding spin-dependent relativistic corrections read (for the case of equal masses $m_1 = m_2 = m$):

- *spin–orbit term*:

$$\boxed{H_{\text{LS}} = \frac{1}{2m^2 r} \mathbf{L} \cdot \mathbf{S} \left[3 \frac{d}{dr} V_{\text{V}}(r) - \frac{d}{dr} V_{\text{S}}(r) \right]}, \quad (2.4)$$

where

$$\mathbf{S} \equiv \mathbf{S}_1 + \mathbf{S}_2$$

is the total spin of the bound state and

$$\mathbf{L} \equiv \mathbf{x} \times \mathbf{p}$$

is the relative orbital angular momentum of its constituents;

- *spin–spin term*:

$$\boxed{H_{\text{SS}} = \frac{2}{3m^2} \mathbf{S}_1 \cdot \mathbf{S}_2 \Delta V_{\text{V}}(r)}; \quad (2.5)$$

- *tensor term*:

$$H_T = \frac{1}{m^2} \left[\frac{(\mathbf{S}_1 \cdot \mathbf{x})(\mathbf{S}_2 \cdot \mathbf{x})}{r^2} - \frac{1}{3} \mathbf{S}_1 \cdot \mathbf{S}_2 \right] \left[\frac{1}{r} \frac{d}{dr} V_V(r) - \frac{d^2}{dr^2} V_V(r) \right] ,$$

or, with the abbreviation

$$S_{12} \equiv 12 \left[\frac{(\mathbf{S}_1 \cdot \mathbf{x})(\mathbf{S}_2 \cdot \mathbf{x})}{r^2} - \frac{1}{3} \mathbf{S}_1 \cdot \mathbf{S}_2 \right] \quad (2.6)$$

of the spin-dependent factor, which is sometimes called the tensor operator,

$$\boxed{H_T = \frac{1}{12 m^2} S_{12} \left[\frac{1}{r} \frac{d}{dr} V_V(r) - \frac{d^2}{dr^2} V_V(r) \right]} . \quad (2.7)$$

The corresponding expressions for the general case of unequal masses of the bound-state constituents $m_1 \neq m_2$ may be obtained in a similar manner; they are collected in Appendix F.

The total angular momentum \mathbf{J} of the respective bound state under consideration—which constitutes, of course, nothing else but the spin of the corresponding composite particle—is clearly given by the sum of

- the relative orbital angular momentum

$$\mathbf{L} \equiv \mathbf{x} \times \mathbf{p}$$

and

- the total spin

$$\mathbf{S} \equiv \mathbf{S}_1 + \mathbf{S}_2$$

of its constituents:

$$\mathbf{J} \equiv \mathbf{L} + \mathbf{S} .$$

Upon squaring this relation,

$$\begin{aligned} \mathbf{J}^2 &= (\mathbf{L} + \mathbf{S})^2 \\ &= \mathbf{L}^2 + \mathbf{S}^2 + 2 \mathbf{L} \cdot \mathbf{S} , \end{aligned}$$

we may express the product $\mathbf{L} \cdot \mathbf{S}$ of relative orbital angular momentum \mathbf{L} and total spin \mathbf{S} in terms of the squares of \mathbf{L} , \mathbf{S} , and \mathbf{J} ,

$$\mathbf{L} \cdot \mathbf{S} = \frac{1}{2} (\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2) ,$$

and, therefore, its expectation values by the corresponding expectation values of \mathbf{L}^2 , \mathbf{S}^2 , and \mathbf{J}^2 :

$$\langle \mathbf{L} \cdot \mathbf{S} \rangle = \frac{1}{2} (\langle \mathbf{J}^2 \rangle - \langle \mathbf{L}^2 \rangle - \langle \mathbf{S}^2 \rangle) .$$

Accordingly, expressed in terms of the quantum numbers ℓ , S , and j of the relative orbital angular momentum \mathbf{L} , the total spin \mathbf{S} , and the total angular momentum \mathbf{J} , respectively, denoting the bound state, the expectation values $\langle \mathbf{L} \cdot \mathbf{S} \rangle$ of the product of orbital angular momentum \mathbf{L} and total spin \mathbf{S} , originating from the spin-orbit term H_{LS} , Eq. (2.4), read

$$\langle \mathbf{L} \cdot \mathbf{S} \rangle = \frac{1}{2} [j(j+1) - \ell(\ell+1) - S(S+1)] .$$

Evidently, the expectation values of the spin-orbit term H_{LS} , Eq. (2.4), vanish for either $\ell = 0$,

$$\langle \mathbf{L} \cdot \mathbf{S} \rangle = 0 \quad \text{for} \quad \ell = 0 ,$$

or $S = 0$,

$$\langle \mathbf{L} \cdot \mathbf{S} \rangle = 0 \quad \text{for} \quad S = 0 ,$$

contributing thus only for $\ell \neq 0$ and $S = 1$. The above relation yields the explicit nonvanishing matrix elements $\langle \mathbf{L} \cdot \mathbf{S} \rangle$ listed in Table 2.1.

Table 2.1: Nonvanishing spin-orbit couplings for $\ell \neq 0$ and $S = 1$

j	$\langle \mathbf{L} \cdot \mathbf{S} \rangle$
$\ell + 1$	ℓ
ℓ	-1
$\ell - 1$	$-(\ell + 1)$

For fermionic bound-state constituents of spin

$$S_1 = S_2 = \frac{1}{2} ,$$

the expectation values $\langle \mathbf{S}_1 \cdot \mathbf{S}_2 \rangle$ of the product of their spins \mathbf{S}_1 and \mathbf{S}_2 in the spin-spin term H_{SS} , Eq. (2.5), have been determined already in Section 1.4:

$$\langle \mathbf{S}_1 \cdot \mathbf{S}_2 \rangle = \begin{cases} -\frac{3}{4} & \text{for spin singlets, i.e., } S = 0 , \\ +\frac{1}{4} & \text{for spin triplets, i.e., } S = 1 . \end{cases}$$

Likewise, for fermionic bound-state constituents of spin

$$S_1 = S_2 = \frac{1}{2} ,$$

the spin-dependent factor S_{12} in the tensor term H_T , Eq. (2.7), may be rewritten as

$$S_{12} = 2 \left[3 \frac{(\mathbf{S} \cdot \mathbf{x})^2}{r^2} - \mathbf{S}^2 \right] . \quad (2.8)$$

Accordingly, the expectation values of the tensor term H_T , Eq. (2.7), also vanish for either $S = 0$,

$$\langle S_{12} \rangle = 0 \quad \text{for} \quad S = 0 ,$$

or $\ell = 0$,

$$\langle S_{12} \rangle = 0 \quad \text{for} \quad \ell = 0 ,$$

the (more or less obvious) reason for the latter being the fact that the rotational symmetry realized in the case of $\ell = 0$ leads to

$$\left\langle \frac{x_i x_j}{r^2} \right\rangle = \frac{1}{3} \delta_{ij} ,$$

which, in turn, results in a mutual cancellation of the two terms on the right-hand side of Eq. (2.8). After a lengthy calculation, the following expression for the *diagonal* matrix elements of S_{12} may be found:

$$\langle S_{12} \rangle = \frac{4}{(2\ell + 3)(2\ell - 1)} \left[\langle \mathbf{S}^2 \rangle \langle \mathbf{L}^2 \rangle - \frac{3}{2} \langle \mathbf{L} \cdot \mathbf{S} \rangle - 3 (\langle \mathbf{L} \cdot \mathbf{S} \rangle)^2 \right] ,$$

which, again only for $\ell \neq 0$ and $S = 1$, yields the explicit nonvanishing expectation values listed in Table 2.2.

Table 2.2: Nonvanishing diagonal matrix elements of S_{12} for $\ell \neq 0$ and $S = 1$

j	$\langle S_{12} \rangle$
$\ell + 1$	$-\frac{2\ell}{2\ell + 3}$
ℓ	2
$\ell - 1$	$-\frac{2(\ell + 1)}{2\ell - 1}$

Chapter 3

The Prototype

We are now in a position where we may start to think seriously about the question of how a realistic, that is, phenomenologically acceptable, potential describing the forces acting between quarks might look like.

3.1 Funnel potential

To begin with, let's summarize our knowledge gained so far. According to the analysis of Section 1.4, the quark–antiquark potential $V_{\text{NR}}(r)$ is of vector and/or scalar type,

$$V_{\text{NR}}(r) = V_{\text{V}}(r) + V_{\text{S}}(r) .$$

For short distances, the potential—arising from one-gluon exchange—is (essentially) Coulomb-like,

$$V_{\text{exch}}(r) = -\frac{4}{3} \frac{\alpha_{\text{s}}}{r} .$$

For large distances, there has to exist a contribution $V_{\text{conf}}(r)$ in order to describe colour confinement,

$$V_{\text{conf}}(r) = a r^n \quad \text{with} \quad n > 0 ,$$

implying that for large r the binding force K must not decrease faster than $1/r$:

$$K = -\frac{d}{dr} V_{\text{conf}}(r) = -n a r^{n-1} = -n \frac{a}{r^{1-n}} .$$

From the mesonic mass spectrum, the exponent n is in the vicinity of

$$n \simeq 1 .$$

For instance, $n = 2$ corresponds to the harmonic oscillator and would lead to equidistant level spacings. Moreover, lattice gauge theories also

find that $V_{\text{NR}}(r)$ is roughly proportional to r for large r . Consequently, a linear potential,

$$V_{\text{conf}}(r) = a r ,$$

is beyond doubt a sensible choice for $V_{\text{conf}}(r)$.

The funnel (or “Coulomb-plus-linear”) potential (Fig. 3.1)

$$V_{\text{NR}}(r) = \underbrace{-\frac{4}{3} \frac{\alpha_s}{r}}_{\text{one-gluon exchange}} + \underbrace{a r}_{\text{confinement}} \quad (3.1)$$

fixed in this way has been the first proposed model [8], which in spite of its simplicity is able to reproduce quite well the charmonium spectrum. In a strict sense, the momentum (-transfer) dependence

$$\alpha_s = \alpha_s(Q^2)$$

of the strong fine structure constant α_s has to be taken into account, modifying thereby the Coulomb-like behaviour of the first term on the right-hand side of Eq. (3.1).

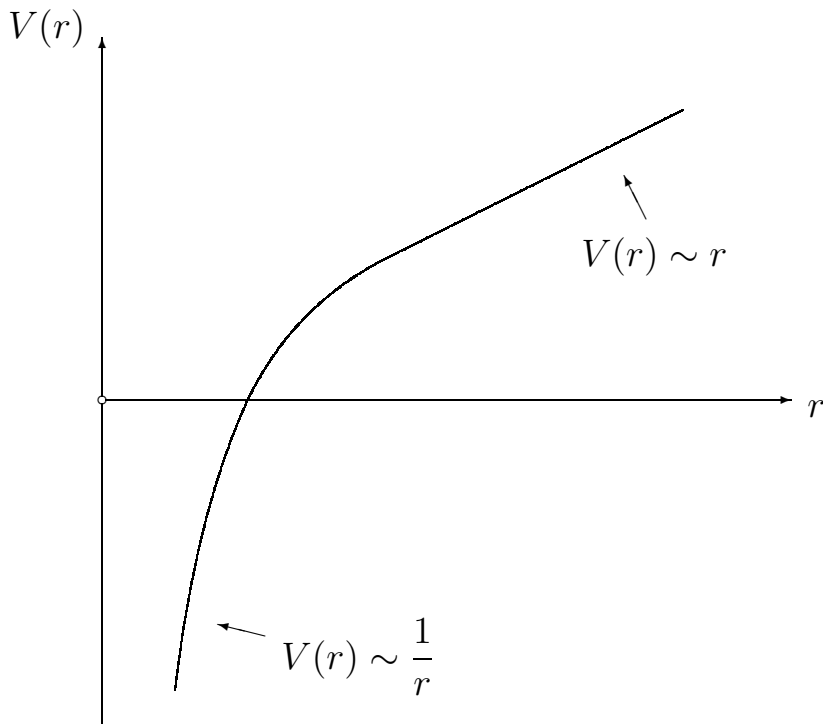


Figure 3.1: Funnel potential.

3.2 Lorentz structure of the funnel potential

In order to decide whether the Lorentz structure of the funnel potential (3.1) is a pure vector, or a pure scalar, or a mixing of both, we consider the P-wave spin splittings of charmonium and bottomonium, that is, the ratio of mass differences [9]

$$\rho = \frac{M(^3P_2) - M(^3P_1)}{M(^3P_1) - M(^3P_0)} . \quad (3.2)$$

(Recall the usual spectroscopic notation, which designates a state with orbital angular momentum ℓ , spin S , and total angular momentum j by $^{2S+1}L_j$, where the capital L \equiv S, P, D, F, \dots represents the orbital angular momentum $\ell = 0, 1, 2, 3, \dots$, respectively.)

Table 3.1: Masses (in GeV) and the ratio ρ [Eq. (3.2)] for the ($\ell = 1, S = 1$) states of charmonium and bottomonium [10]

Level	($c\bar{c}$)	($b\bar{b}$)	($b\bar{b}$)'
3P_0	3.4151	9.8598	10.2321
3P_1	3.5105	9.8919	10.2552
3P_2	3.5562	9.9132	10.2685
ρ	0.478	0.664	0.576

From Table 3.1, the experimental average for ρ [10],

$$\begin{aligned} \rho &= 0.48 & \text{for } (c\bar{c}) , \\ \rho &= 0.66 & \text{for } (b\bar{b}) , \\ \rho &= 0.58 & \text{for } (b\bar{b})' , \end{aligned}$$

is

$$\rho_{\text{exp}} \simeq 0.6 .$$

With the help of the generalized Breit–Fermi Hamiltonian (2.2), we calculate this ratio ρ for the potential (3.1) perturbatively. Since the spin–spin interaction, H_{SS} , does not depend on the total angular momentum, its contribution cancels in a perturbative evaluation of ρ . Accordingly, ρ is determined by the contributions of spin–orbit term, H_{LS} ,

and tensor term, H_T , only:

$$\rho = \frac{\langle {}^3P_2 | H_{LS} + H_T | {}^3P_2 \rangle - \langle {}^3P_1 | H_{LS} + H_T | {}^3P_1 \rangle}{\langle {}^3P_1 | H_{LS} + H_T | {}^3P_1 \rangle - \langle {}^3P_0 | H_{LS} + H_T | {}^3P_0 \rangle} .$$

From Tables 2.1 and 2.2, we find for the expectation values $\langle \mathbf{L} \cdot \mathbf{S} \rangle$ and $\langle S_{12} \rangle$:

$$\langle \mathbf{L} \cdot \mathbf{S} \rangle = \begin{cases} -2 & \text{for } {}^3P_0 , \\ -1 & \text{for } {}^3P_1 , \\ 1 & \text{for } {}^3P_2 , \end{cases}$$

and

$$\langle S_{12} \rangle = \begin{cases} -4 & \text{for } {}^3P_0 , \\ 2 & \text{for } {}^3P_1 , \\ -\frac{2}{5} & \text{for } {}^3P_2 . \end{cases}$$

- For a pure vector, i.e.,

$$V_V = V_{NR} , \quad V_S = 0 ,$$

one obtains

$$\rho = \frac{1}{5} \frac{8 \alpha_s \langle r^{-3} \rangle + 7 a \langle r^{-1} \rangle}{2 \alpha_s \langle r^{-3} \rangle + a \langle r^{-1} \rangle} ,$$

which implies the bounds

$$\frac{4}{5} \leq \rho \leq \frac{7}{5}$$

corresponding to $a = 0$ and $\alpha_s = 0$, respectively, in clear conflict with the experimental finding $\rho_{\text{exp}} \simeq 0.6$.

- A pure scalar, i.e.,

$$V_S = V_{NR} , \quad V_V = 0 ,$$

leads to

$$\rho = 2 ,$$

which also is not tolerable from an experimental point of view.

- A vector/scalar mixing, i.e.,

$$V_V = V_{\text{exch}} = -\frac{4}{3} \frac{\alpha_s}{r} , \quad V_S = V_{\text{conf}} = a r ,$$

results in

$$\rho = \frac{1}{5} \frac{8 \alpha_s \langle r^{-3} \rangle - \frac{5}{2} a \langle r^{-1} \rangle}{2 \alpha_s \langle r^{-3} \rangle - \frac{1}{4} a \langle r^{-1} \rangle} ,$$

which implies

$$\rho \leq \frac{4}{5} ,$$

if the Coulomb part dominates, and

$$\rho \geq 2 ,$$

if the linear part dominates.

Consequently, we arrive at the conclusion that the funnel potential V_{NR} in Eq. (3.1) must be a linear combination of a vector and a scalar part,

$$V_{\text{NR}}(r) = V_{\text{V}}(r) + V_{\text{S}}(r) ,$$

where the Coulomb part V_{exch} is of vector type,

$$V_{\text{V}}(r) = V_{\text{exch}}(r) = -\frac{4}{3} \frac{\alpha_s}{r} ,$$

and the linear part V_{conf} is of scalar type,

$$V_{\text{S}}(r) = V_{\text{conf}}(r) = a r .$$

In summary, from the analysis of the most general conceivable spin structures and the experimentally observed quarkonium mass spectra, we have been able to determine unambiguously the basic shape of the potential acting between quarks:

The interquark potential $V_{\text{NR}}(r) = V_{\text{V}}(r) + V_{\text{S}}(r)$ essentially consists of a Coulomb part V_{exch} , which is of vector type,

$$V_{\text{V}}(r) = V_{\text{exch}}(r) = -\frac{4}{3} \frac{\alpha_s}{r} ,$$

as well as of a linear part V_{conf} , which is of scalar type,

$$V_{\text{S}}(r) = V_{\text{conf}}(r) = a r .$$

In this form, the funnel potential represents the genuine prototype of all “QCD-inspired” potential models proposed for the description of hadrons as bound states of (“constituent”) quarks [1, 2, 3]. A selection of more sophisticated potential models may be found in Appendix G.

Appendix A

S Matrix, Cross-section, and Decay Width

The normalization of creation and annihilation operators is reflected by the (anti-) commutation relations of these operators:

- For the case of bosons, the nonvanishing commutators are

$$[a(\mathbf{p}), a^\dagger(\mathbf{q})] = \delta^{(3)}(\mathbf{p} - \mathbf{q}) .$$

- For the case of fermions, the nonvanishing anticommutators are

$$\{b(\mathbf{p}, \sigma), b^\dagger(\mathbf{q}, \tau)\} = \{d(\mathbf{p}, \sigma), d^\dagger(\mathbf{q}, \tau)\} = \delta^{(3)}(\mathbf{p} - \mathbf{q}) \delta_{\sigma\tau} .$$

Normalizing the vacuum state $|0\rangle$ according to

$$\langle 0|0\rangle = 1 ,$$

the normalizations of the one-particle states read

- for bosons, generically denoted by B,

$$\langle B(\mathbf{p})|B(\mathbf{q})\rangle = \delta^{(3)}(\mathbf{p} - \mathbf{q})$$

and

- for fermions, generically denoted by F,

$$\langle F(\mathbf{p}, \sigma)|F(\mathbf{q}, \tau)\rangle = \delta^{(3)}(\mathbf{p} - \mathbf{q}) \delta_{\sigma\tau} .$$

Let us define, for any transition $i \rightarrow f$ from some initial state i to some final state f , like, for instance, a scattering or decay process, the *S-matrix element* S_{fi} by

$$S_{fi} \equiv \langle f, \text{out} | i, \text{in} \rangle = \langle f | S | i \rangle$$

and the *reduced T-matrix element* T_{fi} by

$$S_{\text{fi}} =: \delta_{\text{fi}} + i (2\pi)^4 \delta^{(4)}(P_{\text{f}} - P_{\text{i}}) T_{\text{fi}} ,$$

where P_{i} and P_{f} denote the total momenta of initial and final state, respectively. The corresponding *transition probability* W_{fi} is the square of the modulus of the transition amplitude $S_{\text{fi}} - \delta_{\text{fi}}$:

$$W_{\text{fi}} \equiv |S_{\text{fi}} - \delta_{\text{fi}}|^2 .$$

For a finite spatial volume V and a finite time interval T , the obscure square of the δ function may be replaced by

$$\begin{aligned} [(2\pi)^4 \delta^{(4)}(P_{\text{f}} - P_{\text{i}})]^2 &= (2\pi)^4 \delta^{(4)}(P_{\text{f}} - P_{\text{i}}) \int d^4x \exp[i (P_{\text{f}} - P_{\text{i}}) x] \\ &= (2\pi)^4 \delta^{(4)}(P_{\text{f}} - P_{\text{i}}) V T , \end{aligned}$$

which leads to

$$W_{\text{fi}} = (2\pi)^4 \delta^{(4)}(P_{\text{f}} - P_{\text{i}}) V T |T_{\text{fi}}|^2 .$$

The *transition rate* R_{fi} is the transition probability per unit time:

$$\boxed{R_{\text{fi}} \equiv \frac{W_{\text{fi}}}{T} = (2\pi)^4 \delta^{(4)}(P_{\text{f}} - P_{\text{i}}) V |T_{\text{fi}}|^2} .$$

The *cross-section* σ_{fi} is the above transition rate R_{fi} divided by the product of the observed flux of the incoming particles, $j = n v_{\text{rel}}$, times the number of target particles, $n V$; here, v_{rel} is the relative velocity of the scattered particles and n denotes generically the particle densities. If necessary, one has to sum over the final states and to average over the initial states, which will be indicated below by a primed sum over the possible spin polarizations σ :

$$\sigma_{\text{fi}} \equiv \frac{1}{v_{\text{rel}} (\prod_{i=1,2} n_i) V} \int \left(\prod_f d^3p_f \right) \sum_{\sigma}' R_{\text{fi}} .$$

With the particle density

$$n = \frac{1}{(2\pi)^3} ,$$

corresponding to a normalization volume of size $(2\pi)^3$, the resulting cross-section $\sigma(\text{i} \rightarrow \text{f})$ for the scattering process $\text{i} \rightarrow \text{f}$ reads

$$\boxed{\sigma(\text{i} \rightarrow \text{f}) = \frac{(2\pi)^{10}}{v_{\text{rel}}} \int \left(\prod_f d^3p_f \right) \delta^{(4)}(P_{\text{f}} - P_{\text{i}}) \sum_{\sigma}' |T_{\text{fi}}|^2} .$$

The product of the energies E_1 and E_2 of the two scattered particles and of their relative velocity v_{rel} forms a Lorentz invariant:

$$E_1 E_2 v_{\text{rel}} = \sqrt{(p_1 p_2)^2 - m_1^2 m_2^2} .$$

The *decay width* Γ_{fi} is the transition rate R_{fi} divided by the number of decaying particles $n V$. If necessary, one again has to sum over the final states and to average over the initial states:

$$\Gamma_{\text{fi}} \equiv \frac{1}{n_{\text{i}} V} \int \left(\prod_f d^3 p_f \right) \sum_{\sigma}' R_{\text{fi}} .$$

The *partial decay width* $\Gamma(\text{i} \rightarrow \text{f})$ for the decay of the particle i into a particular final state f is therefore given by

$$\Gamma(\text{i} \rightarrow \text{f}) = (2\pi)^7 \int \left(\prod_f d^3 p_f \right) \delta^{(4)}(P_{\text{f}} - P_{\text{i}}) \sum_{\sigma}' |T_{\text{fi}}|^2 .$$

The *total decay width* $\Gamma(\text{i})$ of the particle i, which is, of course, nothing else but the inverse of the average lifetime τ_{i} of this particle, is obtained by summing over all possible, kinematically allowed decay channels f:

$$\Gamma(\text{i}) \equiv \frac{1}{\tau_{\text{i}}} = \sum_{\text{f}} \Gamma(\text{i} \rightarrow \text{f}) .$$

Appendix B

Feynman Rules for a General Gauge Theory

First of all, a little *warning*: The correct application of Feynman rules requires some experience. In particular, one clearly should be careful

- when identifying all Feynman diagrams regarded as relevant for the specific process under consideration and
- when computing the combinatorial factors (cf. rule # 3 below).

In order to remain on the safe side, it is advisable to evaluate n -point Green's functions with the help of “Wick's theorem.” Wick's theorem allows to convert time-ordered products of field operators, like those appearing in the S operator

$$S = T \exp \left[i \int d^4x \mathcal{L}_I(x) \right] ,$$

into a sum of products of propagators and normal-ordered products of field operators. A particular Feynman graph is then nothing else but the symbolical representation of a particular operator in the series of the Wick decomposition. S-matrix elements may be obtained from the Green's functions along the course of the “LSZ reduction technique.”

A general unbroken non-Abelian gauge theory for Dirac fermions ψ but without scalar bosons is described in R_ξ gauge by the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} + \bar{\psi} (i \not{D} - m) \psi - \frac{\xi}{2} (\partial_\mu V_\mu^a)^2 + (\partial_\mu \bar{\zeta}) D^\mu \zeta ,$$

$$F_{\mu\nu}^a \equiv \partial_\mu V_\nu^a - \partial_\nu V_\mu^a + g f_{abc} V_\mu^b V_\nu^c ,$$

$$D_\mu \equiv \partial_\mu - i g V_\mu^a T^a .$$

The fermions ψ transform according to an arbitrary representation of the gauge group.

The modification of the original theory brought about by the gauge fixing must be compensated—in order to maintain the unitarity of the S matrix—by adding a further (in general, not Hermitean) term to the Lagrangian, which involves anticommuting, scalar “ghost” fields $\zeta_a, \bar{\zeta}_a$.

For this theory, the complete list of Feynman rules in momentum space for the computation of $i T_{fi}$, where T_{fi} is the reduced T-matrix element defined (in Appendix A) in terms of the S-matrix element S_{fi} by

$$S_{fi} \equiv \langle f, \text{out} | i, \text{in} \rangle =: \delta_{fi} + i (2\pi)^4 \delta^{(4)}(P_f - P_i) T_{fi} ,$$

is:

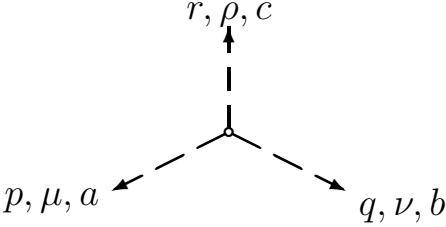
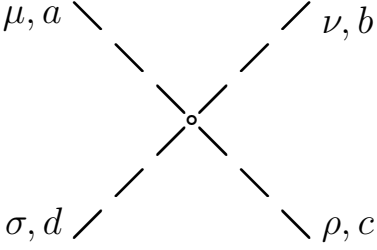
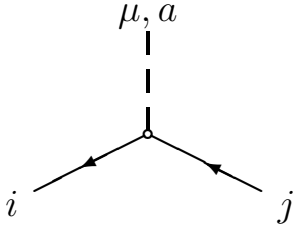
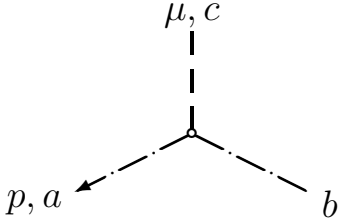
1. Propagators:

Table B.1: Feynman rules for the propagators in a general gauge theory

Vector-boson propagator	$\mathcal{L}_0 = -\frac{1}{4} (\partial_\mu V_\nu^a - \partial_\nu V_\mu^a)^2 - \frac{\xi}{2} (\partial_\mu V_a^\mu)^2$
$\mu, a \quad \text{---} \text{---} \text{---} \text{---} \quad \nu, b$	$= i D_F(k)_{\mu\nu}^{ab} ,$ $D_F(k)_{\mu\nu}^{ab} = \frac{1}{k^2 + i\epsilon} \left[-g_{\mu\nu} + \left(1 - \frac{1}{\xi} \right) \frac{k_\mu k_\nu}{k^2 + i\epsilon} \right] \delta_{ab}$
Fermion propagator	$\mathcal{L}_0 = \bar{\psi} (i \not{\partial} - m) \psi$
$i \longrightarrow j$	$= i S_F(k) ,$ $S_F(k) = \frac{1}{\not{k} - m + i\epsilon} \equiv \frac{\not{k} + m}{k^2 - m^2 + i\epsilon}$
Ghost propagator	$\mathcal{L}_0 = (\partial_\mu \bar{\zeta}_a) \partial^\mu \zeta_a$
$a \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} b$	$= i \Delta_F(k)_{ab} ,$ $\Delta_F(k)_{ab} = \frac{1}{k^2 + i\epsilon} \delta_{ab}$

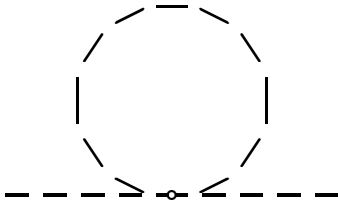
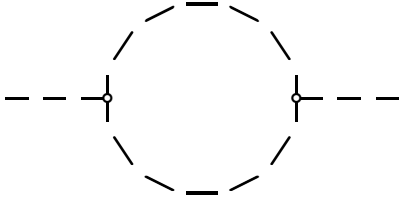
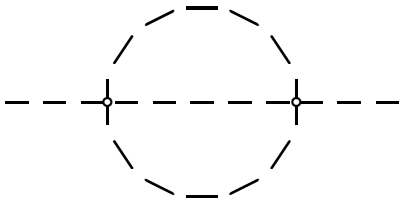
2. Vertices:

Table B.2: Feynman rules for the vertices in a general gauge theory

Three-vector-boson vertex	$\mathcal{L}_I = -g f_{abc} V_a^\mu V_b^\nu \partial_\mu V_\nu^c$
	$= -g f_{abc} [g_{\mu\nu} (p - q)_\rho + g_{\nu\rho} (q - r)_\mu + g_{\rho\mu} (r - p)_\nu]$
Four-vector-boson vertex	$\mathcal{L}_I = -\frac{1}{4} g^2 f_{abc} f_{ade} V_\mu^b V_\nu^c V_d^\mu V_e^\nu$
	$= -i g^2 [f_{abe} f_{cde} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) + f_{ace} f_{bde} (g_{\mu\nu} g_{\rho\sigma} - g_{\mu\sigma} g_{\rho\nu}) + f_{ade} f_{bce} (g_{\mu\nu} g_{\sigma\rho} - g_{\mu\rho} g_{\sigma\nu})]$
Vector-boson-fermion vertex	$\mathcal{L}_I = g \bar{\psi} \not{V} \psi$
	$= i g \gamma_\mu T_{ij}^a$
Vector-boson-ghost vertex	$\mathcal{L}_I = -g f_{abc} (\partial_\mu \bar{\zeta}_a) \zeta_b V_c^\mu$
	$= g f_{abc} p_\mu$

3. *Symmetry factors*: For Feynman graphs involving identical particles in internal lines there arise certain combinatorial factors, which have to be introduced in order to avoid “double counting.” Some examples for this are presented in Table B.3.

Table B.3: Combinatorial factors for Feynman diagrams involving identical particles in internal lines

Feynman graph	Statistical factor
	$\frac{1}{2!}$
	$\frac{1}{2!}$
	$\frac{1}{3!}$

4. For each closed loop of anticommuting fields, i.e., fermions or ghosts, a factor

$$(-1) \text{ .}$$

5. *Loop integration*: At every vertex, energy-momentum conservation has to be taken into account. For every internal and independent four-momentum k , i.e., one which is *not* constrained by energy-momentum conservation at the vertices, an integration

$$\int \frac{d^4 k}{(2\pi)^4} \text{ .}$$

6. *External particles*: Let the polarization vectors $\epsilon_\mu(p, \lambda)$ describing (massless) vector bosons V_μ of four-momentum p and spin polarization λ be normalized according to

$$\sum_\lambda \epsilon_\mu(p, \lambda) \epsilon_\nu^*(p, \lambda) = -g_{\mu\nu} .$$

Let the Dirac spinors $u(p, \sigma)$ and $v(p, \sigma)$ describing fermions ψ of mass m , four-momentum p , and spin polarization σ be normalized according to

$$\begin{aligned} \bar{u}(p, \sigma) u(p, \tau) &= \delta_{\sigma\tau} , \\ \bar{v}(p, \sigma) v(p, \tau) &= -\delta_{\sigma\tau} , \end{aligned}$$

which is equivalent to

$$u^\dagger(p, \sigma) u(p, \tau) = v^\dagger(p, \sigma) v(p, \tau) = \frac{E_p}{m} \delta_{\sigma\tau} ,$$

where

$$E_p \equiv \sqrt{\mathbf{p}^2 + m^2} .$$

The above normalization implies for the energy projection operators

$$\begin{aligned} \sum_\sigma u(p, \sigma) \bar{u}(p, \sigma) &= \frac{\not{p} + m}{2m} , \\ \sum_\sigma v(p, \sigma) \bar{v}(p, \sigma) &= \frac{\not{p} - m}{2m} . \end{aligned}$$

With the above conventions, the expansions of the corresponding field operators in terms of plane waves read

- for Hermitean vector bosons

$$\begin{aligned} V_\mu(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{\sqrt{2E_p}} \sum_\lambda [a(p, \lambda) \epsilon_\mu(p, \lambda) \exp(-i p x) \\ + a^\dagger(p, \lambda) \epsilon_\mu^*(p, \lambda) \exp(i p x)] \end{aligned}$$

and

- for Dirac fermions

$$\begin{aligned} \psi(x) = \frac{1}{(2\pi)^{3/2}} \int d^3p \sqrt{\frac{m}{E_p}} \sum_\sigma [b(p, \sigma) u(p, \sigma) \exp(-i p x) \\ + d^\dagger(p, \sigma) v(p, \sigma) \exp(i p x)] , \end{aligned}$$

$$\begin{aligned} \bar{\psi}(x) = \frac{1}{(2\pi)^{3/2}} \int d^3p \sqrt{\frac{m}{E_p}} \sum_\sigma [d(p, \sigma) \bar{v}(p, \sigma) \exp(-i p x) \\ + b^\dagger(p, \sigma) \bar{u}(p, \sigma) \exp(i p x)] . \end{aligned}$$

Table B.4: Feynman rules for the external particles in a general gauge theory

Particle	incoming	outgoing
Vector boson	$\frac{1}{(2\pi)^{3/2} \sqrt{2} E_p} \epsilon_\mu(p, \lambda)$	$\frac{1}{(2\pi)^{3/2} \sqrt{2} E_p} \epsilon_\mu^*(p, \lambda)$
Fermion	$\frac{1}{(2\pi)^{3/2}} \sqrt{\frac{m}{E_p}} u(p, \sigma)$	$\frac{1}{(2\pi)^{3/2}} \sqrt{\frac{m}{E_p}} \bar{u}(p, \sigma)$
Antifermion	$\frac{1}{(2\pi)^{3/2}} \sqrt{\frac{m}{E_p}} \bar{v}(p, \sigma)$	$\frac{1}{(2\pi)^{3/2}} \sqrt{\frac{m}{E_p}} v(p, \sigma)$

7. For each change of the relative order of external fermions, a factor

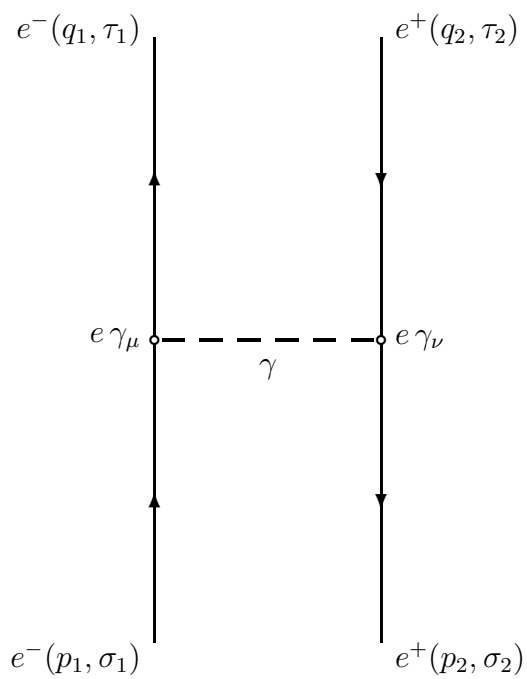
$$(-1) \text{ .}$$

For illustrative purposes, let's consider electron–positron scattering in lowest non-trivial order of the perturbative, that is, loop, expansion (cf. Fig. B.1):

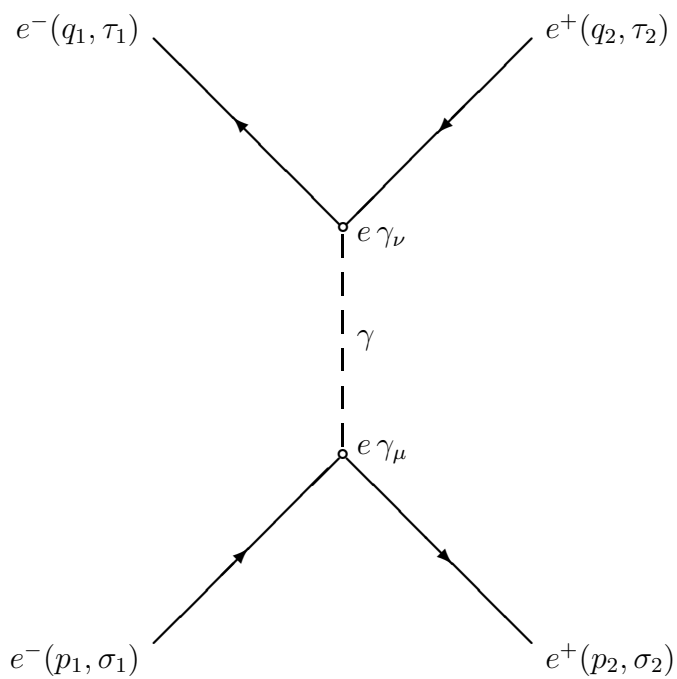
$$e^-(p_1, \sigma_1) + e^+(p_2, \sigma_2) \rightarrow e^-(q_1, \tau_1) + e^+(q_2, \tau_2) \text{ .}$$

Applying the above Feynman rules, we may immediately write down the corresponding scattering amplitude T_{fi} (in the so-called Feynman gauge, defined by fixing the gauge parameter ξ to the value $\xi = 1$):

$$\begin{aligned}
T_{\text{fi}} = & - \frac{1}{(2\pi)^6} \frac{m^2}{\sqrt{E_{p_1} E_{p_2} E_{q_1} E_{q_2}}} e^2 \\
& \times \left[\frac{1}{(p_1 - q_1)^2} \bar{u}(q_1, \tau_1) \gamma_\mu u(p_1, \sigma_1) \bar{v}(p_2, \sigma_2) \gamma^\mu v(q_2, \tau_2) \right. \\
& \left. - \frac{1}{(p_1 + p_2)^2} \bar{u}(q_1, \tau_1) \gamma_\mu v(q_2, \tau_2) \bar{v}(p_2, \sigma_2) \gamma^\mu u(p_1, \sigma_1) \right] \text{ .}
\end{aligned}$$



(a)



(b)

Figure B.1: Electron–positron scattering in lowest order of the perturbation expansion: (a) one-photon exchange, (b) pair annihilation.

Appendix C

Quantum Chromodynamics

Quantum chromodynamics (QCD) is that quantum field theory which is generally believed to describe the strong interactions. It is the special case of a general gauge theory, characterized by the following features:

- The gauge group is $SU(3)_C$ (where C stands for *colour*), describing the (unbroken) symmetry acting on the colour degrees of freedom. The order or dimension of the Lie group $SU(N)$ is $N^2 - 1$, which equals 8 in the case of $SU(3)$. Hence, $SU(3)$ has eight generators T^a , $a = 1, 2, \dots, 8$. Of course, the totally antisymmetric structure constants f_{abc} , $a, b, c = 1, 2, \dots, 8$, are those of $SU(3)$; their values may be found in Appendix D.1.
- The particle content of QCD comprises the following vector-boson and fermion fields:
 - *Vector bosons*: There are eight, of course, massless, gluons G_μ^a transforming according to the adjoint, i.e., eight-dimensional, representation of $SU(3)$.
 - *Fermions*: There are at least six quarks $q_f = u, d, s, c, b, t, \dots$, each of them transforming according to the fundamental, i.e., three-dimensional, representation of $SU(3)$. The generators of $SU(3)$ in the fundamental representation are given by

$$T_{\text{fund}}^a = \frac{\lambda_a}{2} ,$$

where λ_a , $a = 1, 2, \dots, 8$, label the eight Gell-Mann matrices; an explicit representation of the latter matrices may be found in Appendix D.2. The total number of quark flavours will be denoted by n_F : $f = 1, 2, \dots, n_F$.

Consequently, the Lagrangian defining QCD reads

$$\begin{aligned}\mathcal{L}^{\text{QCD}} &= -\frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} + \sum_{f=1}^{n_F} \bar{q}_f (i \not{D} - m_f) q_f \\ &+ \text{ gauge-fixing terms} \\ &+ \text{ ghost terms } ,\end{aligned}$$

with the gluon field strength

$$F_{\mu\nu}^a = \partial_\mu G_\nu^a - \partial_\nu G_\mu^a + g_s f_{abc} G_\mu^b G_\nu^c$$

and the gauge-covariant derivative acting on the quark fields

$$D_\mu = \partial_\mu - i g_s G_\mu^a \frac{\lambda_a}{2} .$$

The parameters of this theory are the strong coupling constant g_s and the (current) quark masses m_f .

Appendix D

SU(3)

D.1 Structure constants

The Lie group SU(3) is defined by the following commutation relations for its eight generators T^a , $a = 1, \dots, 8$:

$$[T^a, T^b] = i f_{abc} T^c ,$$

where the nonvanishing elements among the structure constants f_{abc} are listed in Table D.1.

Table D.1: Nonvanishing structure constants f_{abc} of SU(3)

a	b	c	f_{abc}
1	2	3	1
1	4	7	1/2
1	5	6	-1/2
2	4	6	1/2
2	5	7	1/2
3	4	5	1/2
3	6	7	-1/2
4	5	8	$\sqrt{3}/2$
6	7	8	$\sqrt{3}/2$

In addition, the generators T^a satisfy the anticommutation relations

$$\{T^a, T^b\} = \frac{1}{3} \delta_{ab} + d_{abc} T^c ,$$

where the nonvanishing elements among the coefficients d_{abc} are listed in Table D.2.

Table D.2: Nonvanishing coefficients d_{abc} of SU(3)

a	b	c	d_{abc}	a	b	c	d_{abc}
1	1	8	$1/\sqrt{3}$	3	5	5	$1/2$
1	4	6	$1/2$	3	6	6	$-1/2$
1	5	7	$1/2$	3	7	7	$-1/2$
2	2	8	$1/\sqrt{3}$	4	4	8	$-1/(2\sqrt{3})$
2	4	7	$-1/2$	5	5	8	$-1/(2\sqrt{3})$
2	5	6	$1/2$	6	6	8	$-1/(2\sqrt{3})$
3	3	8	$1/\sqrt{3}$	7	7	8	$-1/(2\sqrt{3})$
3	4	4	$1/2$	8	8	8	$-1/\sqrt{3}$

The structure constants f_{abc} are totally antisymmetric whereas the coefficients d_{abc} are totally symmetric under permutations of indices.

D.2 Gell-Mann matrices

In the fundamental, i.e., three-dimensional, representation of SU(3), the eight generators T^a , $a = 1, \dots, 8$, are explicitly given by

$$T_{\text{fund}}^a = \frac{\lambda_a}{2} ,$$

where λ_a are the eight Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} ,$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} ,$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} , \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} ,$$

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} .$$

D.3 Traces

The traces of the simplest products of Gell-Mann matrices read

$$\begin{aligned}
 \text{Tr}(\lambda_a) &= 0 \ , \\
 \text{Tr}(\lambda_a \lambda_b) &= 2 \delta_{ab} \ , \quad \sum_a \text{Tr}(\lambda_a^2) = 16 \ , \\
 \text{Tr}(\lambda_a [\lambda_b, \lambda_c]) &= 4 i f_{abc} \ , \\
 \text{Tr}(\lambda_a \{\lambda_b, \lambda_c\}) &= 4 d_{abc} \ , \\
 \text{Tr}(\lambda_a \lambda_b \lambda_c) &= 2 i f_{abc} + 2 d_{abc} \ .
 \end{aligned}$$

Some further useful relations are:

$$\begin{aligned}
 \sum_{a,b,c} (f_{abc})^2 &= 24 \ , \\
 \sum_{a,b,c} (d_{abc})^2 &= \frac{40}{3} \ , \\
 \sum_{j,k,a} \epsilon_{ijk} \frac{\lambda_{\ell j}^a}{2} \frac{\lambda_{km}^a}{2} &= -\frac{2}{3} \epsilon_{ilm} \ .
 \end{aligned}$$

Appendix E

$$\nabla_i \nabla_j \Phi(r)$$

We would like to express the second derivatives $\nabla_i \nabla_j \Phi(r)$ with respect to Cartesian coordinates $\mathbf{x} \equiv \{x_1, x_2, x_3\}$ of an arbitrary function $\Phi(r)$ which depends merely on the radial coordinate $r \equiv |\mathbf{x}|$ in terms of the “spherically symmetric” derivatives coming into question, that is, the first and second derivatives of Φ with respect to r —which we indicate by prime(s)—as well as the Laplacian $\Delta \equiv \nabla \cdot \nabla$ of Φ . In other words, we would like to rewrite these second derivatives $\nabla_i \nabla_j \Phi(r)$ in terms of $\Phi'(r)$, $\Phi''(r)$, and $\Delta\Phi(r)$. To this end, we start from the most general ansatz for the expression we are looking for, viz., from

$$\begin{aligned} \nabla_i \nabla_j \Phi(r) &= \left(a \delta_{ij} + b \frac{x_i x_j}{r^2} \right) \frac{1}{r} \Phi'(r) \\ &+ \left(c \delta_{ij} + d \frac{x_i x_j}{r^2} \right) \Phi''(r) \\ &+ \left(e \delta_{ij} + f \frac{x_i x_j}{r^2} \right) \Delta\Phi(r) , \end{aligned}$$

where, for every term, the powers of the radial coordinate r have been chosen in such a way that the coefficients a, b, \dots, f are dimensionless.

It is a simple and straightforward task to determine the coefficients a, b, \dots, f :

- On the one hand, we contract the above ansatz by multiplying it by δ_{ij} and by summing over i and j . Using

$$\delta_{ij} \delta_{ij} = \delta_{ij} \delta_{ji} = \text{Tr}(1_{3 \times 3}) = 3 ,$$

we obtain

$$\Delta\Phi(r) = (3a + b) \frac{1}{r} \Phi'(r) + (3c + d) \Phi''(r) + (3e + f) \Delta\Phi(r) .$$

By comparing both sides of this equation, we may conclude that the coefficients a, b, \dots, f have to satisfy the relationships

$$3a + b = 0 ,$$

$$3c + d = 0 ,$$

$$3e + f = 1 ,$$

which, in turn, imply

$$a = -\frac{b}{3} ,$$

$$c = -\frac{d}{3} ,$$

$$e = \frac{1}{3} - \frac{f}{3} .$$

Consequently, taking into account these relations and combining corresponding terms, our ansatz simplifies to

$$\begin{aligned} \nabla_i \nabla_j \Phi(r) &= \left(\frac{x_i x_j}{r^2} - \frac{1}{3} \delta_{ij} \right) \left[b \frac{1}{r} \Phi'(r) + d \Phi''(r) + f \Delta \Phi(r) \right] \\ &+ \frac{1}{3} \delta_{ij} \Delta \Phi(r) . \end{aligned}$$

- On the other hand, for the case $i \neq j$, we may easily calculate any second derivative $\nabla_i \nabla_j \Phi(r)$ explicitly. The first derivative $\nabla_i \Phi(r)$ of $\Phi(r)$ with respect to any of the Cartesian coordinates x_i reads

$$\nabla_i \Phi(r) = \frac{x_i}{r} \Phi'(r) .$$

Consequently, for $i \neq j$, the second derivatives $\nabla_i \nabla_j \Phi(r)$ of $\Phi(r)$ are given by

$$\nabla_i \nabla_j \Phi(r) = \frac{x_i x_j}{r^2} \left[\Phi''(r) - \frac{1}{r} \Phi'(r) \right] \quad \text{for } i \neq j ,$$

whereas, by gaining advantage from the fact that $\delta_{ij} = 0$ for $i \neq j$, the above, already simplified expression for $\nabla_i \nabla_j \Phi(r)$ reduces to

$$\nabla_i \nabla_j \Phi(r) = \frac{x_i x_j}{r^2} \left[b \frac{1}{r} \Phi'(r) + d \Phi''(r) + f \Delta \Phi(r) \right] \quad \text{for } i \neq j .$$

The comparison of these two expressions allows us to fix the three, until now indeterminate coefficients b , d , and f :

$$b = -1 ,$$

$$d = +1 ,$$

$$f = 0 .$$

In summary, upon collecting all our findings, the second derivatives $\nabla_i \nabla_j \Phi(r)$ may be expressed as the following linear combinations of the “spherically symmetric” derivatives $\Phi'(r)$, $\Phi''(r)$, and $\Delta\Phi(r)$:

$$\boxed{\nabla_i \nabla_j \Phi(r) = \left(\frac{x_i x_j}{r^2} - \frac{1}{3} \delta_{ij} \right) \left[\Phi''(r) - \frac{1}{r} \Phi'(r) \right] + \frac{1}{3} \delta_{ij} \Delta\Phi(r) } .$$

Appendix F

Some Further Formulae for Spectroscopy

For unequal masses $m_1 \neq m_2$, the various spin-dependent relativistic corrections to a nonrelativistic potential

$$V_{\text{NR}}(r) = V_V(r) + V_S(r)$$

of vector-plus-scalar Lorentz structure read

- for the spin-orbit term

$$\begin{aligned} H_{\text{LS}} = & \frac{1}{4 m_1^2 m_2^2} \frac{1}{r} \left\{ \left[((m_1 + m_2)^2 + 2 m_1 m_2) \mathbf{L} \cdot \mathbf{S}_+ \right. \right. \\ & + \left. (m_2^2 - m_1^2) \mathbf{L} \cdot \mathbf{S}_- \right] \frac{d}{dr} V_V(r) \\ & - \left. \left[(m_1^2 + m_2^2) \mathbf{L} \cdot \mathbf{S}_+ + (m_2^2 - m_1^2) \mathbf{L} \cdot \mathbf{S}_- \right] \frac{d}{dr} V_S(r) \right\} , \end{aligned}$$

with

$$\mathbf{S}_+ \equiv \mathbf{S}_1 + \mathbf{S}_2$$

and

$$\mathbf{S}_- \equiv \mathbf{S}_1 - \mathbf{S}_2 ;$$

- for the spin-spin term

$$H_{\text{SS}} = \frac{2}{3 m_1 m_2} \mathbf{S}_1 \cdot \mathbf{S}_2 \Delta V_V(r) ;$$

and

- for the tensor term

$$H_{\text{T}} = \frac{1}{12 m_1 m_2} S_{12} \left[\frac{1}{r} \frac{d}{dr} V_V(r) - \frac{d^2}{dr^2} V_V(r) \right] .$$

The signatures—*parity* P , *charge conjugation* C , and *G parity*—for a quark–antiquark bound state with relative orbital angular momentum ℓ , spin S , and isospin I are given by

$$\begin{aligned} P(q\bar{q}) &= (-1)^{\ell+1} , \\ C(q\bar{q}) &= (-1)^{\ell+S} , \\ G(q\bar{q}) &= (-1)^{\ell+S+I} . \end{aligned}$$

For instance, for the pion we have $\ell = 0$, $S = 0$, $I = 1$ and therefore

$$\begin{aligned} P(\pi) &= -1 , \\ C(\pi) &= +1 , \\ G(\pi) &= -1 . \end{aligned}$$

Appendix G

Various Potential Models

Eichten et al. [8]	$V_{\text{NR}} = -\frac{4}{3} \frac{\alpha_s}{r} + a r$
Quigg–Rosner [11]	$V_{\text{NR}} = A \ln(r/r_0)$
Richardson [12]	$V_{\text{NR}} = -\frac{4}{3} \frac{48 \pi^2}{33 - 2 n_{\text{F}}} \frac{1}{(2\pi)^3} \int d^3 q \frac{\exp(i \mathbf{q} \cdot \mathbf{x})}{\mathbf{q}^2 \ln(1 + \mathbf{q}^2/\Lambda^2)}$
Ono–Schöberl [13]	$V_{\text{NR}} = -b \exp(-r/c) + \begin{cases} -\frac{4}{3} \frac{\alpha_s}{r} + d & \text{for } r \leq R_1 \\ a r & \text{for } r \geq R_1 \end{cases}$ $R_1 = \sqrt{\frac{4 \alpha_s}{3 a}} , \quad \delta^{(3)}(\mathbf{x}) \rightarrow \frac{1}{4\pi r_0^2} \frac{\exp(-r/r_0)}{r}$ $V_{\text{S}} = a r , \quad V_{\text{V}} = V_{\text{NR}} - V_{\text{S}}$
Martin [14]	$V_{\text{NR}} = A + B r^{0.1}$
Buchmüller et al. [15]	$V_{\text{NR}} = -\frac{4}{3} \frac{1}{(2\pi)^3} \int d^3 q \exp(i \mathbf{q} \cdot \mathbf{x}) \frac{4\pi \alpha_s(\mathbf{q}^2)}{\mathbf{q}^2}$
Falkensteiner et al. [16]	$V_{\text{NR}} = -\frac{4}{3} \frac{\alpha_s}{r} \text{erf}(\sqrt{\pi} A r) + a r$

$$\text{Schöberl [17]} \quad V_{\text{NR}} = -b \exp(-r/c) + \begin{cases} -\frac{4}{3} \frac{\alpha_s}{r} \operatorname{erf}(A r) + d & \text{for } r \leq R_1 \\ a r & \text{for } r \geq R_1 \end{cases}$$

$$V_{\text{S}} = a r, \quad V_{\text{V}} = V_{\text{NR}} - V_{\text{S}}$$

$$\begin{aligned} \text{Flamm et al. [18]} \quad V_{\text{V}} &= -\frac{4}{3} \frac{\alpha_s}{(r + r_0)^{1.107}} (1 - c) + a r^{0.91} (1 - d) \\ V_{\text{S}} &= -\frac{4}{3} \frac{\alpha_s}{(r + r_0)^{1.107}} c + a r^{0.91} d \\ V_{\text{NR}} &= V_{\text{V}} + V_{\text{S}} \end{aligned}$$

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